

# Chapter 3

## DISPERSION AND MIXING

### 3.1 Shear Dispersion

#### *Shear flow*

In the preceding sections, we restricted our attention to advection by uniform flows. While a uniform flow causes a mere translation of the pollutant (transport without distortion), a non-uniform flow can produce new and important effects. The cause is the differential advection: Different parts of the fluid flow at different rates, bringing into close proximity fluid parcels with different histories and thus relatively independent concentrations. This intensifies concentration gradients and promotes diffusion. At the extreme, a turbulent flow, by its very irregular structure, is highly favorable to mixing.

A simpler case than turbulence but one where diffusion is nonetheless greatly enhanced is that of a shear flow. In a shear flow, the velocity varies in the transverse direction, so that parcels on different flow lines travel at different speeds, the faster ones overtaking the slower ones. The prototypical example is the uniform shear, where the velocity varies linearly in the transverse direction [ $u(z) = a + bz$ ].

In this chapter, we denote the transverse direction by  $z$  (instead of  $y$ , the alphabetical choice) because environmental flows are typically sheared in the vertical. Examples are rivers (where the current is weaker at depth because of bottom friction), lakes and oceans (where the flow is stronger near the surface under wind action), and the lower atmosphere (where winds are weaker near the ground and increase with height – Have you ever flown a kite?).

#### *Dispersion*

Dispersion is simply another word for diffusion; but with a specific meaning. We use the word dispersion when we describe a process that appears as

diffusion but does not exactly proceed according to Fick's law (i.e., with a flux proportional to the concentration gradient).

*The shear-dispersion process*

The necessary ingredients for this process to occur are:

1. existence of a shear flow, say  $u(z)$ ,
2. diffusion in the direction transverse to the flow, and
3. presence of a substance (contaminant) that is carried and diffused.

In two dimensions (with  $x$  and  $z$  as the streamwise and transverse directions, respectively), the concentration distribution of a substance that is both carried differentially by the flow and diffused transversely is governed by the equation:

$$\frac{\partial c}{\partial t} + u(z) \frac{\partial c}{\partial x} = D \frac{\partial^2 c}{\partial z^2}. \quad (3.1)$$

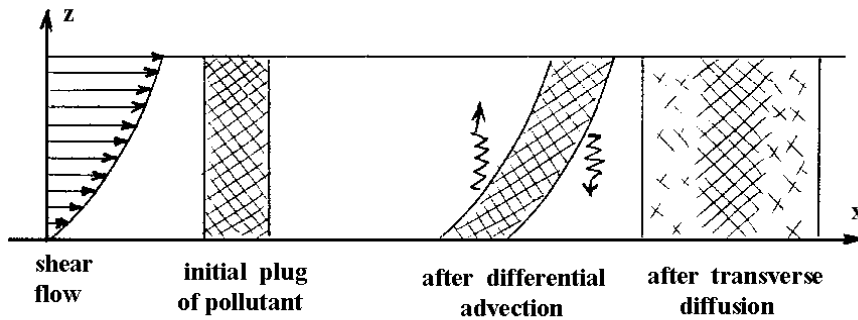
We may neglect diffusion in the streamwise direction (the  $D\partial^2 c/\partial x^2$  term) by restricting our attention to cases with a large Peclet number (see Section 2-8),

$$Pe = \frac{UL}{D} \gg 1. \quad (3.2)$$

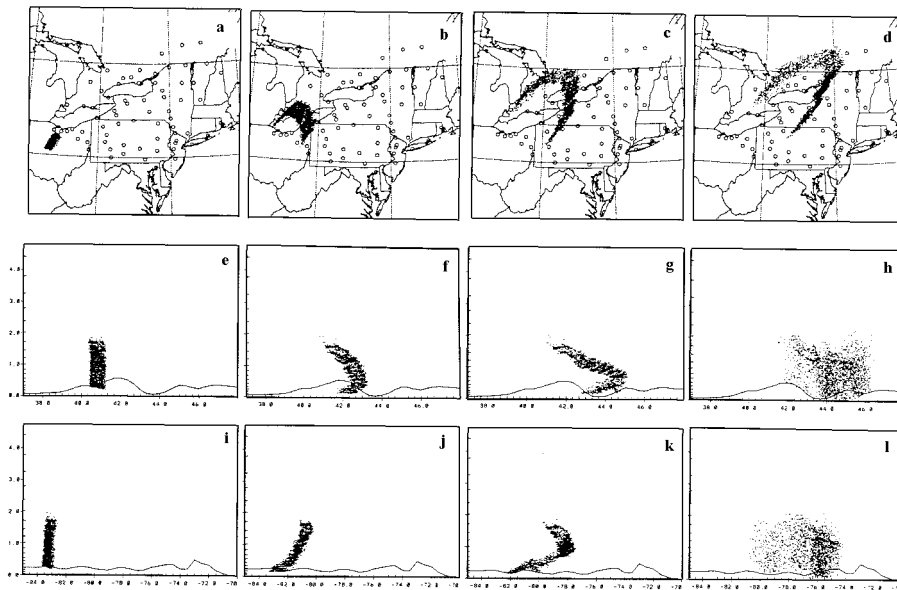
We shall validate this assumption at a later stage. Here,  $U$  is the scale for  $u(z)$ , such as its maximum or averaged absolute value, and  $L$  is the downstream length scale, such as length of the contaminated area.

To anticipate what happens, let us imagine that advection and diffusion act alternatively over short periods of time. We then arrive at the scenario depicted in Figure 3-1: The plug of contaminant is differentially advected and distorted, transverse diffusion then smears cross-stream differences in concentration, and the new plug of contaminant has grown wider in the downstream direction. Consequently, the combined effect of differential advection and transverse diffusion is longitudinal spreading (that is, spreading along the flow).

A clear example of shear-induced dispersion is depicted in Figure 3-2. A numerical model of the weather over the northeastern United States is seeded with a cloud of passive particles during a run for the conditions existing during 25–26 September 1983. By design, the initial cloud is spatially concentrated (in northern Ohio) and extends from the surface upward to 2000 m. The simulation begins at night (11 p.m.) when the lower atmosphere is stratified (i.e., vertical turbulence is very weak and the wind is significantly varying with height). The result is differential advection during the night and early morning of the next day. The plug of particles undergoes great distortions. Then, solar radiation during the morning, which is absorbed not by the atmosphere but by the opaque ground, gradually raises the ground temperature. The lower atmosphere is heated from below, and vertical convection sets in. The result is an enhanced vertical turbulence and reduction in the vertical wind shear. Rapid vertical diffusion redistributes the particles over the vertical, and by 5 p.m. the group of particles has achieved vertical homogeneity, with the net effect of being much wider than it was initially.



**Figure 3-1.** Sequential depiction of the processes that act simultaneously during shear dispersion. Note how differential movement and transverse diffusion combine to cause longitudinal spreading.



**Figure 3-2.** Instantaneous particle clouds for travel times of 6, 12, 18 and 24 hours on 25–26 September 1983 according to a meteorological model. The top panels [a to d] are plan views, middle panels [e to h] are side views looking westward, and the bottom panels [i to l] are side views looking northward. Note the shear-induced distortions during the non-turbulent hours of the night and morning, followed by vertical diffusion once vertical convection has developed. The net effect is a horizontal spreading of the initial cloud – the shear dispersion effect. [From Moran and Pielke, 1994]

The preceding example nicely illustrates the process of shear dispersion, by separating in time its two ingredients. Almost always, however, differential advection and transverse diffusion act not alternatively but simultaneously. Nonetheless, we can expect a very analogous outcome.

The question that arises naturally is whether the longitudinal spreading caused by the combined action of differential advection and transverse diffusion is similar to the spreading caused by a diffusive process. In other words, can shear-induced longitudinal dispersion be represented by Fick's law of diffusion? The answer to this question is affirmative, in a limit: Longitudinal diffusion due to the shear effect obeys the law of Fickian diffusion when transverse diffusion is sufficiently rapid.

*Random-walk analogue*

Let us illustrate the preceding assertion by a simple, numerical example using the random-walk concept. Recall the 1D random-walk model of Section 2-2, but now take three levels, to include a transverse direction. To represent the shear flow, make the top row move one box to the right at each time step, the middle row not move, and the bottom row move one box to the left at each time step. Finally, mimic rapid transverse diffusion by averaging numbers vertically at each step. As an example, consider the following sequence:

<b>initial state,</b>	0	0	3	12	18	12	3	0	0
<b>vertically</b>	0	0	3	12	18	12	3	0	0
<b>homogenized</b>	0	0	3	12	18	12	3	0	0
<b>action of</b>	0	0	0	3	12	18	12	3	0
<b>differential</b>	0	0	3	12	18	12	3	0	0
<b>advection</b>	0	3	12	18	12	3	0	0	0
<b>subsequent action</b>	0	1	5	11	14	11	5	1	0
<b>of transverse</b>	0	1	5	11	14	11	5	1	0
<b>diffusion</b>	0	1	5	11	14	11	5	1	0

The combined effect of differential advection and transverse diffusion is the replacement of the shorter sequence (3, 12, 18, 12, 3) by the longer one (1, 5, 11, 14, 11, 5, 1). The very same result could have been obtained directly by the rule that one third of the particles move one box to the left, one third to the right, and the last third remain in place. Such a rule is a restatement of the random-walk process, albeit with different probabilities than those in the example constructed earlier (Section 2-2). Since the random-walk process is known to be similar to that of diffusion, we can conclude that the longitudinal spreading by combined differential advection and rapid transverse diffusion is a form of diffusion.

It follows that longitudinal spreading can be represented mathematically by a term of the form:

$$K \frac{\partial^2 c}{\partial x^2},$$

where  $K$  is the effective diffusion coefficient that results from the combined differential advection [ $u(z)$ ] and transverse diffusion [ $D$ ]. The immediate question is: What is the value of  $K$ ? Or, how does  $K$  vary with  $u(z)$  and  $D$ ? In the following two subsections, we will present two separate answers, the first more qualitative and yielding only a rough estimate, the second more quantitative and leading to an precise formula for  $K$ . [Note: Do not confuse this  $K$  diffusivity with the decay rate  $K$  of the preceding chapters.]

*Estimation of the effective diffusion*

In this section, a heuristic argument is used to establish an approximate value for the effective diffusivity  $K$ . For an exact determination, see the following sub-section.

The physical description given above relied in part on vertical homogenization under the action of transverse diffusion. And, we have a way of determining how long it takes for the pollutant to diffuse vertically across the system. If  $H$  is the height of the domain, the time  $T$  over which vertical mixing occurs is given by (2.31):

$$T \sim 0.134 \frac{H^2}{D},$$

where the symbol  $\sim$  means ‘must be proportional to’ or ‘is on the order of’. (Note the dimensional consistency.)

Now, the horizontal spread is related to how far apart two fluid parcels are sheared away during this time interval. If we introduce  $\Delta U$  as the scale for the velocity shear ( $\Delta U$  may be taken as  $u_{\max} - u_{\min}$ , and is generally not the average velocity,  $U$ ), two parcels with velocities differing by  $\Delta U$  become separated over the time interval  $T$  by the distance

$$L = \Delta U T \sim 0.134 \frac{H^2 \Delta U}{D}.$$

This is a measure of the horizontal spread of the pollutant patch (Figure 3-1). For an equivalent, effective diffusivity  $K$ , the same spread would be given by the  $4\sigma$ -rule:

$$L \sim 4\sqrt{2KT} \sim 4\sqrt{0.268 \frac{KH^2}{D}}.$$

Equating both estimates of  $L$  yields the relation

$$0.134 \frac{H^2 \Delta U}{D} \sim 4\sqrt{0.268 \frac{KH^2}{D}},$$

which, solved for  $K$ , provides our answer:

$$K \sim 0.0042 \frac{H^2 \Delta U^2}{D}. \quad (3.3)$$

[Exercise: Check the dimensional consistency of this relation.]

Let us now discuss a few properties of the preceding result. That  $K$  is inversely proportional to  $D$ , rather than directly proportional to it, is counter-intuitive. Indeed, we can be inclined to think that the greater the transverse diffusion, the greater the longitudinal dispersion. Not so! It actually works this way: A large  $D$  implies an efficient vertical mixing, which tends to erase (smear) the effect of differential advection; pollutant particles migrate up and down so fast that they essentially all move at the mean speed of the flow, and the shear is unimportant, causing only a weak longitudinal spreading. At the other extreme, a small value of  $D$  implies a long time for vertical exchanges and thus ample time for differential advection to take effect; the pollutant patch is highly distorted while it diffuses moderately in the transverse direction, and longitudinal dispersion is large.

In the absence of shear, isotropic diffusion would cause horizontal diffusion at the rate  $D$ . (The term  $D\partial^2c/\partial x^2$  then needs to be retained in the equation.) We can naturally neglect this term if the new term ( $K\partial^2c/\partial x^2$ ) representing shear dispersion is much larger, namely if  $K \gg D$ . From the preceding analysis, we can anticipate this to be the case when

$$0.0042 \frac{H^2\Delta U^2}{D} \gg D,$$

that is when

$$\frac{H\Delta U}{D} \gg 15. \quad (3.4)$$

In other words, longitudinal dispersion by shear effect overwhelms pure longitudinal diffusion when  $\Delta U$  far exceeds  $D/H$ , where  $\Delta U$  is the magnitude of the shear flow ( $u_{\max} - u_{\min}$ ),  $D$  is the diffusion coefficient, and  $H$  is the transverse width (usually height) of the system. Note how the ratio in (3.4) resembles the Peclet-number definition (2.63).

Condition (3.4) is easily met in practice. Take for example a 10-m deep river where the velocity varies from zero at the bottom to 10 cm/s at the surface ( $\Delta U = 0.1$  m/s). If the turbulence-enhanced diffusivity is  $D = 0.005$  m<sup>2</sup>/s, then

$$\frac{H\Delta U}{D} = 200 \gg 15.$$

Similarly in the lower atmosphere, a typical wind shear varies from nearly zero at the surface to values on the order of 10 m/s at altitudes of several hundred meters, while a typical turbulence-induced diffusivity is 1 m<sup>2</sup>/s. This leads to a value for the ratio in (3.4) easily exceeding 1000. In both cases, longitudinal diffusion is largely dominated by shear dispersion.

#### *Taylor's theory*

We now proceed with a more quantitative approach in order to obtain a formula providing the effective diffusivity from the velocity shear and the transverse diffusion coefficient. The underlying theory is due to G.I. Taylor,

a famous British fluid dynamicist, who first elucidated the shear-dispersion effect in the 1920s. For the sake of mathematical simplicity, we will sketch the elements of the theory, leaving a rigorous derivation to be found in the literature by the interested reader.

Because our primary interest resides in the horizontal spreading of the substance, more than on the details of its vertical distribution, we define the vertical averages of the velocity and concentration, respectively:

$$\bar{u} = \frac{1}{H} \int_0^H u(z) dz, \quad \bar{c} = \frac{1}{H} \int_0^H c dz, \quad (3.5)$$

where  $H$  is the height of the domain. We then define their departures from the mean values:

$$u = \bar{u} + u', \quad c = \bar{c} + c'.$$

The variable  $u'$  contains the velocity-shear information crucial to the dispersion process. The variable  $c'$ , by contrast, represents the inhomogeneities of concentration in the vertical which can be assumed to be much smaller than the average  $\bar{c}$ . Using this notation, we transform Equation (3.1) into

$$\frac{\partial \bar{c}}{\partial t} + \frac{\partial c'}{\partial t} + \bar{u} \frac{\partial \bar{c}}{\partial x} + \bar{u} \frac{\partial c'}{\partial x} + u' \frac{\partial \bar{c}}{\partial x} + u' \frac{\partial c'}{\partial x} = D \frac{\partial^2 c'}{\partial z^2}, \quad (3.6)$$

where we anticipate the  $c'$ -terms to be small compared to the corresponding  $\bar{c}$ -terms. The vertical average of this equation is:

$$\frac{\partial \bar{c}}{\partial t} + \bar{u} \frac{\partial \bar{c}}{\partial x} + \frac{1}{H} \int_0^H u' \frac{\partial c'}{\partial x} dz = \left[ D \frac{\partial c'}{\partial z} \right]_0^H = 0. \quad (3.7)$$

The right-hand side is taken as zero since there is generally no flux of the substance along the lateral boundaries ( $z = 0$  and  $z = H$ ). In (3.6), the first term represents the rate of accumulation in the average, the second advection by the mean flow, and the third the shear effect (via  $u'$ ). This last term is important since we anticipate the shear effect to be the crux of the process. We also hope to transform this term into  $-K \partial^2 \bar{c} / \partial x^2$ .

For Equation (3.6) to be a 3-way balance among its terms, we expect the term  $u' \partial c' / \partial x$  to be as important as  $\partial \bar{c} / \partial t$  and  $\bar{u} \partial \bar{c} / \partial x$ . Yet, as we already assumed, the  $c'$ -terms should be small next to the corresponding  $\bar{c}$ -terms (such as  $u' \partial c' / \partial x \ll u' \partial \bar{c} / \partial x$ ). The only possible ranking of the terms in the left-hand side of (3.5) that meets both requirements is as follows:

$$\begin{aligned} \text{large:} & \quad u' \frac{\partial \bar{c}}{\partial x}, \\ \text{smaller:} & \quad \frac{\partial \bar{c}}{\partial t}, \quad \bar{u} \frac{\partial \bar{c}}{\partial x}, \quad u' \frac{\partial c'}{\partial x}, \\ \text{even smaller:} & \quad \frac{\partial c'}{\partial t}, \quad \bar{u} \frac{\partial c'}{\partial x}. \end{aligned}$$

Finally, we need to estimate the relative size of the term on the right-hand side,  $D\partial^2 c'/\partial z^2$ . This is the term that represents transverse diffusion, which acts together with shear advection to produce the dispersion effect. Furthermore, the term  $u'\partial\bar{c}/\partial x$  is, up to now, unmatched. For these two reasons, we conclude that the term  $D\partial^2 c'/\partial z^2$  must be placed in the top category. All in all, the leading balance in Equation (3.5) is:

$$u' \frac{\partial \bar{c}}{\partial x} = D \frac{\partial^2 c'}{\partial z^2}.$$

Because  $u'$  is a known (given) function of  $z$  and  $\bar{c}$  is  $z$ -independent by definition, we can integrate this equation once and then once more, to obtain successively:

$$\begin{aligned} \frac{\partial c'}{\partial z} &= \frac{1}{D} \left[ \int_0^z u'(z') dz' \right] \frac{\partial \bar{c}}{\partial x}, \\ c' &= \frac{1}{D} \left[ \int_0^z dz' \int_0^{z'} u'(z'') dz'' \right] \frac{\partial \bar{c}}{\partial x} + \text{const.} \end{aligned}$$

The first constant of integration is set to zero to enforce the no-flux condition at  $z = 0$ . Requiring that  $c'$  has no vertical average (so that the average is all in  $\bar{c}$ ), we determine the remaining constant of integration and obtain

$$c' = \frac{1}{D} \left[ \int_0^z dz' \int_0^{z'} u'(z'') dz'' - \frac{1}{H} \int_0^H dz \int_0^z dz' \int_0^{z'} u'(z'') dz'' \right] \frac{\partial \bar{c}}{\partial x}.$$

Taking the  $x$ -derivative of this expression and substituting in the integral of (3.6), we arrive at an equation for the mean concentration:

$$\frac{\partial \bar{c}}{\partial t} + \bar{u} \frac{\partial \bar{c}}{\partial x} = \left[ \frac{-1}{DH} \int_0^H u'(z) dz \int_0^z dz' \int_0^{z'} u'(z'') dz'' \right] \frac{\partial^2 \bar{c}}{\partial x^2}.$$

This can be written as an advection-diffusion equation for  $\bar{c}$ :

$$\frac{\partial \bar{c}}{\partial t} + \bar{u} \frac{\partial \bar{c}}{\partial x} = K \frac{\partial^2 \bar{c}}{\partial x^2}, \quad (3.8)$$

if the diffusivity  $K$  is defined as

$$K = \frac{-1}{DH} \int_0^H u'(z) dz \int_0^z dz' \int_0^{z'} u'(z'') dz''.$$

The introduction of a ‘cumulated velocity shear’

$$\tilde{u}(z) = \frac{1}{H} \int_0^z u'(z') dz' \quad (3.9)$$

and an integration by parts turn the integrals of  $K$  into a more succinct expression:



$$K = \frac{H}{D} \int_0^H \tilde{u}^2(z) dz, \quad (3.10)$$

which also shows that this effective diffusivity is always positive. Since  $K \geq 0$  for all  $u(z)$  functions, we conclude that the shear effect is dispersive for all possible shear-velocity profiles. Furthermore, by definition  $\tilde{u}$  scales as  $\Delta U$ , and, in turn, expression (3.9) shows that  $K$  is on the order of  $H^2 \Delta U^2 / D$ . This confirms our earlier reasoning, which led to the qualitative estimate (3.3).

For the linear velocity profile

$$u(z) = u_0 + \Delta U \frac{z}{H},$$

the effective diffusivity is found to be

$$K = \frac{1}{120} \frac{H^2 \Delta U^2}{D}. \quad (3.11)$$

## 3.2 Turbulent Dispersion

### *Turbulent flow*

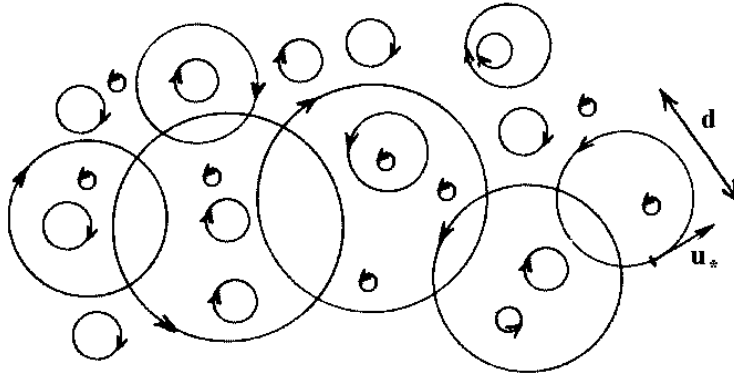
Almost all environmental fluid flows (of air and water) are turbulent. Therefore, the existing agitation that effectively mixes the fluid and diffuses pollutants is expected to be much greater than the underlying molecular agitation. This leads to an effective value of the diffusion coefficient  $D$  much larger than its molecular value.

The situation is not unlike that of shear flows, because erratic motions in the turbulent fluid can be thought of as many shear flows superimposed on one another. Unfortunately, the analysis of shear-flow dispersion cannot be extended to turbulent flows because, unlike a simple shear flow where the velocity profile is known, the highly intermittent and irregular character of turbulent motions defies description. In fact, there does not exist yet a unifying theory of turbulence, not even one for the turbulent statistical properties. So, the approach will necessarily be much more empirical and heuristic.

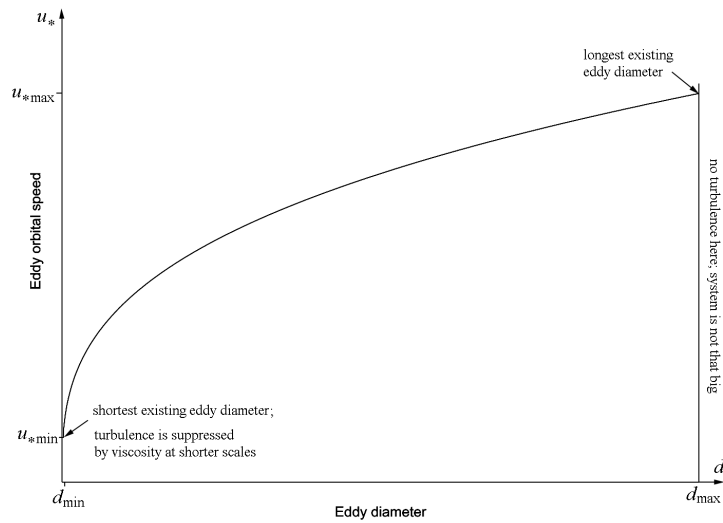
In a statistical description of turbulence, two variables play a fundamental role. These are (Figure 3-3):

$u_*$  = characteristic orbital velocity of fluid parcels in the turbulent eddies, and  $d$  = characteristic diameter of these eddies.

Actually, the turbulent fluid is populated by many eddies, of varying sizes and speeds, and as a result  $u_*$  and  $d$  do not assume each of a single value but vary within a certain realizable range. In stationary, homogeneous and isotropic turbulence (that is, a turbulent flow that statistically appears unchanging in time, uniform in space and without preferential direction), all eddies of a given size (same  $d$ ) behave more or less in the same way and thus share the same characteristic velocity  $u_*$ . In other words, we expect  $u_*$  to be a function of  $d$  (Figure 3-4).



**Figure 3-3.** Kolmogorov's depiction of a turbulent flow, which consists in the superposition of many simultaneous eddies of various sizes and intensities.



**Figure 3-4.** The eddy orbital velocity versus the eddy length scale. The largest eddies spin the fastest.

From the pair of quantities  $(d, u_*)$  emerges a third one, an intrinsic time scale. It is defined as:

$$\tau(d) = \frac{\pi d}{u_*(d)} \sim \frac{d}{u_*(d)}, \quad (3.12)$$

and is called the eddy turn-around time, for it is the time taken by a particle to cover the circumference  $\pi d$  at the speed  $u_*$  (nominal orbital velocity of eddy with diameter  $d$ ).

From the quantities  $u_*$ ,  $d$  and  $\tau$ , we can estimate a diffusion coefficient  $D(d)$  that characterizes the diffusion by eddies of size  $d$ . We can arrive at the same result in three different ways.

1. Dimensional analysis: Since the dimensions of  $D(d)$  must be length squared per time, its expression in terms of  $d$  and  $u_*$  can only be proportional to their product, i.e.

$$D(d) \sim du_*(d), \quad (3.13)$$

where the symbol  $\sim$  means ‘must be proportional to’ or ‘is on the order of’. By virtue of (3.9), using the pair of variables  $(d, \tau)$  or  $(u_*, \tau)$  would have led to the same result.

2. Diffusive spreading: Under diffusion governed by diffusivity  $D(d)$ , a patch grows in time according to (2.17), namely

$$\text{patch size} = 4\sigma = 4\sqrt{2D(d)t}.$$

In half an eddy turn-around time  $\tau(d)$ , a pollutant particle moves one diameter  $d$  away, and thus the patch size must be  $d$  by time  $\tau(d)/2$ :

$$d = 4\sqrt{2D(d)\frac{\tau(d)}{2}} = 4\sqrt{D(d)\frac{\pi d}{u_*(d)}}.$$

Using (3.10) to eliminate  $\tau(d)$  and solving for  $D(d)$ , we obtain

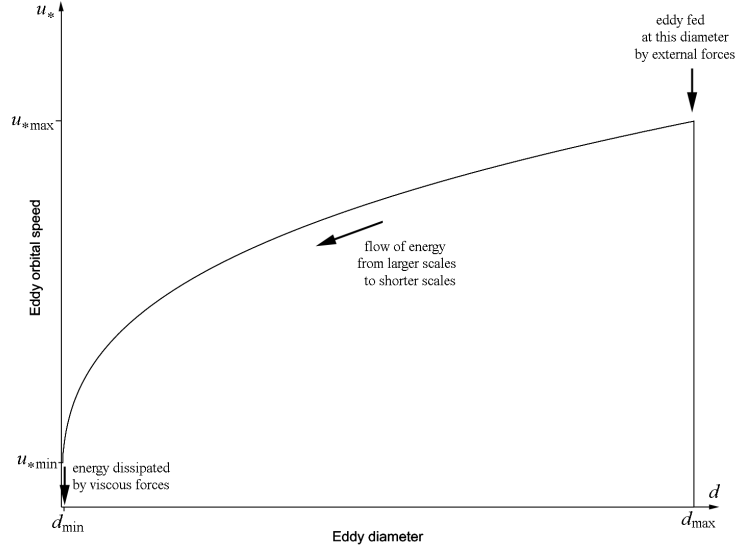
$$D(d) = \frac{1}{16\pi} du_*(d) = 0.020 du_*(d) \sim du_*(d).$$

3. Random – walk analogy: If we match eddies of size  $d$  with bins of width  $\Delta x$ , we ought to take the eddy turn-around time as the corresponding time step:  $d = \Delta x$ ,  $\tau(d) = \Delta t$ . Then, the random-walk model yields the diffusion coefficient [recall (2.14)]:

$$D(d) = \frac{\Delta x^2}{4\Delta t} = \frac{d^2}{4\tau(d)} \sim du_*(d).$$

We again recover (3.11).

Therefore, all our considerations lead us to state that the turbulent diffusion coefficient caused by eddies of a given size is proportional to the product of the eddy diameter with the corresponding characteristic orbital velocity.



**Figure 3-5.** The turbulent energy cascade. According to this theory, the energy fed by external forces excites the largest possible eddies and is passed gradually to ever smaller eddies, all the way to a minimum scale where this energy is dissipated by viscosity.

#### *Large versus small eddies*

Turbulent motions naturally involve many eddies of various sizes and intensities, all embedded in one another. So, we must consider a continuous range of values for  $u_*$  and  $d$ . The question then arises as to which particular values should be used to construct the overall diffusion coefficient  $D$ . The answer is the pair of values  $(d, u_*)$  that maximizes  $D(d)$ , because the operating diffusion (the one with the greatest influence) corresponds to the fastest spreading and thus to the largest value of  $D(d)$ .

According to the theory established in the 1940s by A.N. Kolmogorov, turbulent motions span a wide range of scales ranging from a macroscale at which the energy is supplied, to a microscale at which energy is dissipated by viscosity. The interaction among the eddies at the various scales passes energy from the larger eddies gradually to the smaller ones. This process is known as the turbulent energy cascade (Figure 3-5).

If the state of turbulence is statistically steady (statistically unchanging turbulence intensity), then the rate of energy transfer from one scale to the next must be the same for all scales, so that no particular scale (= group of eddies at that scale) see its energy level increase or decrease over time. It follows that the rate at which energy is supplied at the largest possible scale ( $d_{max}$ ) is equal to that dissipated at the shortest scale ( $d_{min}$ ). Let us denote by  $\epsilon$  this rate of energy supply/dissipation, per unit mass of fluid:

- $\epsilon$  = energy supplied to fluid per unit mass and time
- = energy cascading from scale to scale, per unit mass and time
- = energy dissipated by viscosity, per unit mass and time.

The dimensions of  $\epsilon$  are:

$$[\epsilon] = \frac{ML^2T^{-2}}{MT} = L^2T^{-3}. \quad (3.14)$$

With Kolmogorov, we further assume that the characteristics of the turbulent eddies of scale  $d$  depend solely on  $d$  itself (the eddies know how big they are) and on the energy cascade rate (the eddies know at which rate energy is supplied to them and at which rate they must supply it to the next smaller eddies in the cascade). Thus  $u_*$  depends only on  $d$  and  $\epsilon$ . Since  $[u_*] = LT^{-1}$ ,  $[d] = L$  and  $[\epsilon] = L^2T^{-3}$ , the only possibility is:

$$u_*(d) \sim (\epsilon d)^{1/3}. \quad (3.15)$$

Laboratory experiments suggest that the coefficient of proportionality is close to one:  $u_*(d) \simeq 0.95 (\epsilon d)^{1/3}$ . The value for this coefficient, however, is poorly known, for it may be affected by a number of factors, including the manner by which the turbulent energy is fed in the fluid, the shape of the domain, etc.

Discussion. The larger  $\epsilon$ , the larger  $u_*$ . This makes sense, for the more energy that is supplied to the system, the more vigorous the eddies are. The smaller  $d$ , the weaker  $u_*$ . This could not have been anticipated and must be considered as a result. The implication is that the largest eddies have the largest speeds, and the smallest ones have the smallest speeds. Thus, turbulent intensity decreases with decreasing length scale.

Now that we know how  $u_*$  depends on  $d$ , we can also determine how the characteristic time scale,  $\tau(d)$ , and the diffusion coefficient,  $D(d)$ , vary with the eddy length scale. Equations (3.9) and (3.10) provide, respectively,

$$\tau(d) = \frac{\pi d}{u_*(d)} \sim \frac{d}{(\epsilon d)^{1/3}} \rightarrow \tau(d) \sim \epsilon^{-1/3} d^{2/3}, \quad (3.16)$$

$$D(d) \sim du_*(d) \sim d(\epsilon d)^{1/3} \rightarrow D(d) \sim \epsilon^{1/3} d^{4/3}. \quad (3.17)$$

Both quantities increase with the eddy size.

As stated earlier, the effective diffusivity of the turbulent flow is the maximum value of  $D(d)$  over all possible values of  $d$ . Since  $D(d)$  is an increasing function of  $d$ , that value is obtained for the largest possible length scale,  $d_{\max}$ . We thus have

$$\begin{aligned} D &= \max[D(d)] \\ &\sim \epsilon^{1/3} d_{\max}^{4/3} \\ &\sim d_{\max} u_*(d_{\max}). \end{aligned} \quad (3.18)$$

In other words, the largest eddies regulate the rate of dispersion or, put the other way, turbulent dispersion is primarily effected by the largest eddies. By contrast, the molecular processes (viscosity), which affect only the shortest scales, are the least effective. Since in most turbulent flows  $d_{\max}$  is much larger than  $d_{\min}$ , the diffusivity caused by turbulence far exceeds the molecular diffusion coefficient.

#### *Length scales*

a) Longest scale: Typically, the largest possible eddies are those that extend over the entire system and therefore

$$d_{\max} = L,$$

where  $L$  is the length scale of the system (such as the width of the domain). In natural flows, there is usually a noticeable scale disparity between a relatively short vertical extent (depth, height) and a relatively long horizontal extent (distance, length) of the system. Examples are:

$$\begin{aligned} \text{rivers :} & \quad \text{depth} \ll \text{width, length} \\ \text{atmosphere :} & \quad \text{height} \ll \text{physically relevant horizontal distances.} \end{aligned}$$

In such situations, we must clearly distinguish between eddies that rotate in the vertical plane (about a horizontal axis) and those that rotate horizontally (about a vertical axis). In rivers, we must furthermore distinguish the transverse eddies from the longitudinal eddies.

Vertical diffusion (bottom-up and top-down mixing) is accomplished by eddies that bring fluid up from below and down from above, i.e. those that rotate in the vertical plane. The corresponding  $d_{\max}$  is then  $H$ , the height of the system. The corresponding  $u_*(H)$  is the scale for the velocity shear (difference between maximum and minimum values of the horizontal velocity). Horizontal diffusion is accomplished by horizontal eddies, and the corresponding  $d_{\max}$  is the width or length of the system (depending on the direction of interest).

b) Shortest scale: As noted earlier, the shortest eddy scale is set by viscosity, because the shorter the eddy scale, the more important is the fluid's viscosity. Consequently, the shortest eddy scale can be defined as the length scale at which viscosity is dominant. Viscosity is, by definition, the molecular diffusion of linear momentum; it is traditionally denoted by  $\nu$ , and its dimensions are identical to those of a diffusivity:

$$[\nu] = [D] = L^2T^{-1}.$$

At the lowest scale, the turbulent diffusion of momentum reduces to viscosity and thus

$$\nu \sim \min[D(d)] \sim \epsilon^{1/3} d_{\min}^{4/3}.$$

Solving for  $d_{\min}$ , we obtain

$$d_{\min} \sim \nu^{3/4} \epsilon^{-1/4}. \quad (3.19)$$

Therefore,  $d_{\min}$  depends on the energy level of the turbulence. The greater the turbulence (the bigger  $\epsilon$ ), the shorter is the finest eddy scale. Typically  $d_{\min}$  is on the order of a few millimeters or shorter. The spread of turbulent length scales can be quantified by the ratio  $d_{\max}/d_{\min}$  and related to the Reynolds number  $Re$  of the flow:

$$\frac{d_{\max}}{d_{\min}} \sim \frac{L}{\nu^{3/4} \epsilon^{-1/4}} \sim \frac{L}{\nu^{3/4} (U^3/L)^{-1/4}} = \left( \frac{UL}{\nu} \right)^{3/4} = Re^{3/4}$$

with the Reynolds number defined as  $Re = UL/\nu$  with  $U \sim (\epsilon L)^{1/3}$  being the velocity associated with the largest vortices.

Note: From here on, to simplify the notation, we shall reserve the notation  $d$ ,  $u_*$  and  $\tau$  to denote the turbulent characteristics of the largest eddies in the turbulent flow under consideration [ $d = d_{\max}$ ,  $u_* = u_*(d_{\max})$ ,  $\tau = \tau(d_{\max})$ ].

### 3.3 Mixing in Homogeneous Fluids

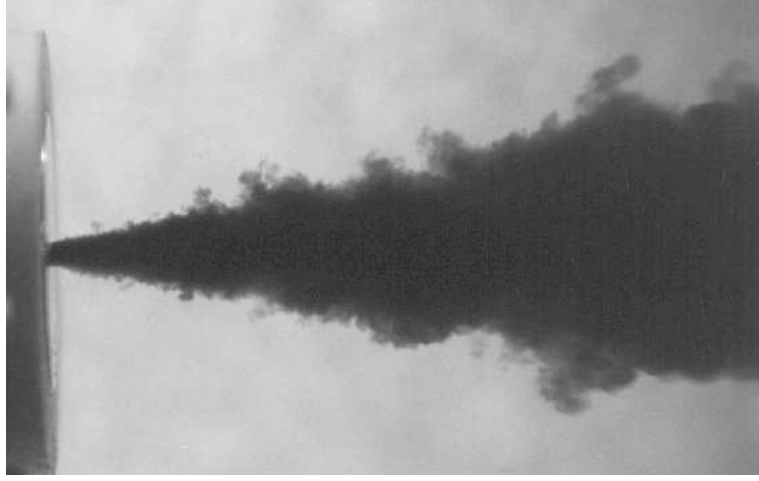
#### *Cause of mixing*

Mixing in fluids, such as the atmosphere or natural bodies of water, is generated by turbulence that is induced either mechanically or thermally, or both. Relegating the problem of mixing in stratified fluids to a later section (Section 3-4), we shall first consider here mixing in the absence of buoyancy forces, i.e. in homogeneous fluids. In such fluids, turbulence arises from shear in the flow, that is the variation of the flow speed in a direction transverse to it. Examples of environmental significance are the jets at the exit of underwater sewage release pipes or at the top of factory smokestacks, a wind blowing over the water surface, and the encounter of flows with different velocities.

Since the properties of a turbulent flow greatly depend on the geometry of the flow domain and on the type of forces acting on the fluid, almost every situation is a separate problem requiring specific investigation. We shall therefore limit ourselves here to the most common case, that of a jet penetrating in an otherwise quiescent fluid.

#### *Mixing caused by a jet*

Laboratory investigations of jets penetrating into a quiescent fluid of the same density (e.g., Figure 3-6) consistently reveal that the envelope containing the turbulence caused by the jet has a nearly conical shape. In other words, the radius  $R$  of the jet is proportional to the distance  $x$  downstream from the discharge location. Further, the opening angle is almost invariably  $11.8^\circ$  (that is approximately  $24^\circ$  from side to opposite side), and therefore the coefficient of proportionality is  $\tan(11.8^\circ) \simeq 1/5$ :



**Figure 3-6.** A water jet emerging from a nozzle into an otherwise undisturbed tank of water. The jet water is colored to be made visible.

$$R = \frac{1}{5} x. \quad (3.20)$$

[Note: Since the initial jet radius is not zero but the finite nozzle radius, equal to half the exit diameter  $d$ , the distance  $x$  must be counted from a distance  $5d/2$  into the inlet pipe. This point of origin is called the *virtual source*.]

Observations suitably averaged over the many turbulent fluctuations reveal that the velocity in the jet obeys a law of similarity: All cross-sections appear identical, except for a stretching factor, and the velocity profile across the jet exhibits a nearly Gaussian shape (bell curve). Therefore, we can write:

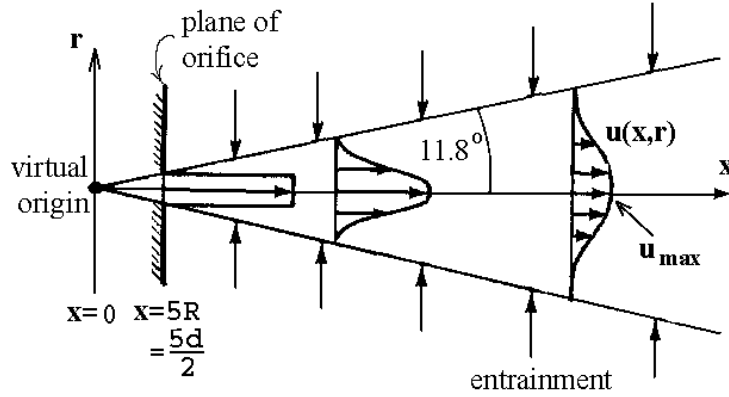
$$u(x, r) = u_{\max} \exp\left(-\frac{r^2}{2\sigma^2}\right)$$

where  $x$  is the downstream distance along the jet (counted from the virtual source),  $r$  is the cross-jet radial distance from its centerline,  $u_{\max}(x)$  is the maximum speed at the centerline, and  $\sigma(x)$  is the standard deviation related to the spread of the profile across the centerline. Since  $4\sigma$  is the width of the distribution [see Equation (2.20)] and since we know it to be the diameter  $2R$  of the jet, we can write  $4\sigma = 2R$ , *i.e.*  $\sigma = x/10$ , which leads to:

$$u(x, r) = u_{\max} \exp\left(-\frac{50r^2}{x^2}\right). \quad (3.21)$$

When a jet enters a fluid at rest, the sole source of momentum is that of the jet itself, and the absence of external accelerating or decelerating forces implies that the momentum flux in the jet's cross-section remains constant downstream. Since this flux is the momentum per unit volume,  $\rho u$  (where  $\rho$  is





**Figure 3-7.** Schematic description of a jet penetrating in a fluid at rest. The widening is linear with distance, and all cross-jet velocity profiles, except those very near the orifice, are similar to one another, after averaging over the turbulent fluctuations.

the fluid density and  $u$  the velocity), times the velocity  $u$  itself cumulated over the jet's cross-section, the statement that momentum is constant downstream is:

$$\int_0^{\infty} \rho u^2 2\pi r dr = \rho U^2 \frac{\pi d^2}{4},$$

where  $U$  and  $d$  are respectively the average exit velocity and the orifice diameter, which are usually known. After calculating the integral and by virtue of (3.18), we deduce:

$$u_{\max} = \frac{5d}{x} U. \quad (3.22)$$

In other words, the velocity along the centerline of the jet decreases inversely with distance from the virtual source (Figure 3-7). To this maximum velocity corresponds an average velocity  $\bar{u}$  defined by

$$\bar{u} = \frac{1}{\pi R^2} \int_0^{\infty} u 2\pi r dr = \frac{u_{\max}}{2} = \frac{5d}{2x} U. \quad (3.23)$$

The volumetric flux  $Q$  is not constant along the jet because of entrainment of quiescent surrounding fluid. It can be calculated as follows

$$Q = \int_0^{\infty} u 2\pi r dr = \frac{\pi}{50} u_{\max} x^2 = \frac{\pi}{10} d U x,$$

and is found to increase linearly with distance. The entrainment rate can be defined as the rate at which the volumetric flux grows with distance, namely

$$E = \frac{dQ}{dx} = \frac{\pi dU}{10} .$$

When the jet contains a contaminant and the ambient fluid does not, this entrainment naturally causes dilution and the contaminant's concentration decreases downstream. Assuming that the concentration profile across the jet is a Gaussian (bell) curve similar to that for the velocity, we write:

$$c(x, r) = c_{\max} \exp\left(-\frac{r^2}{2\sigma^2}\right) = c_{\max} \exp\left(-\frac{50r^2}{x^2}\right) , \quad (3.24)$$

where  $c_{\max}(x)$  is the peak concentration along the centerline, a function of the distance  $x$ . Conservation of the total amount of contaminant transported by the jet (assuming that the ambient fluid is free of any contaminant) requires:

$$\int_0^\infty cu \, 2\pi r dr = c_o U \frac{\pi d^2}{4} ,$$

where  $c_o$  is the average concentration at the orifice. Calculation of the integral provides the manner by which the centerline concentration varies along the jet:

$$c_{\max} = \frac{5d}{x} c_o, \quad (3.25)$$

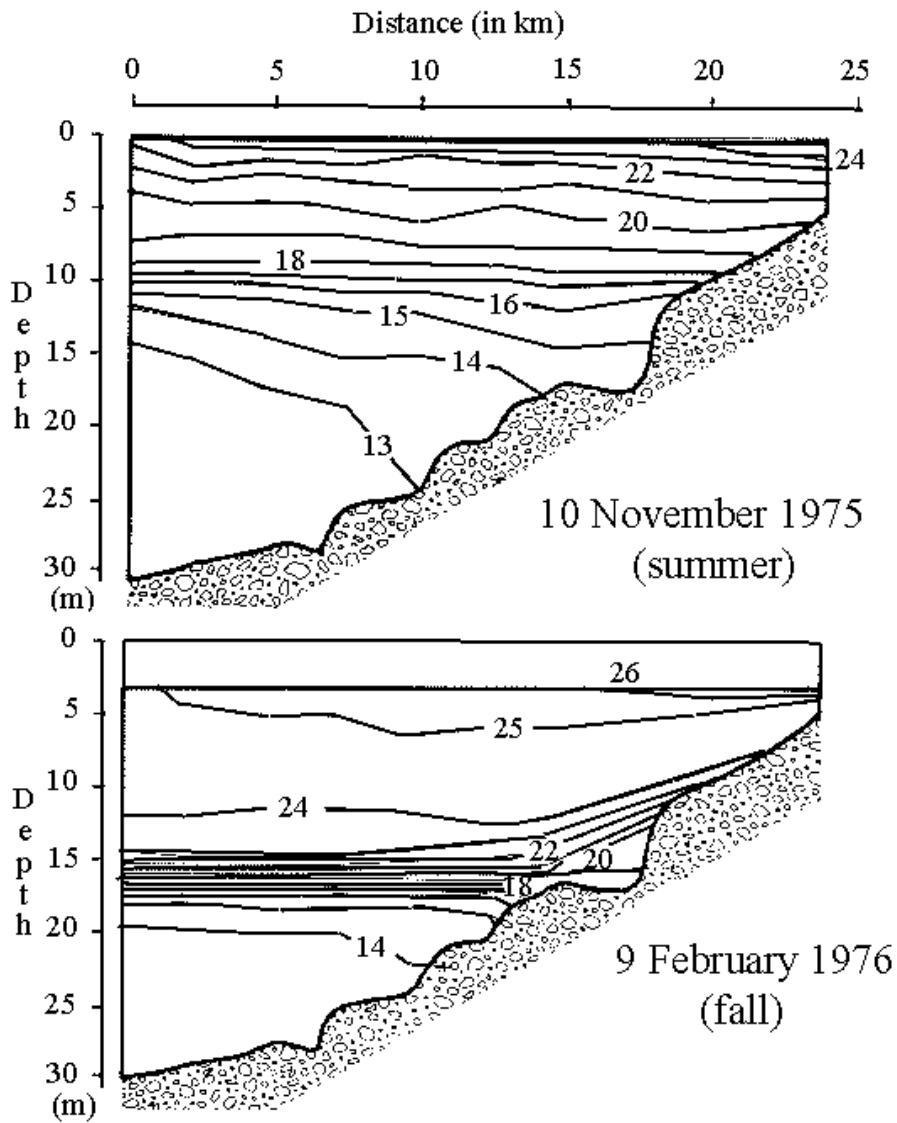
Not surprisingly, because of the dilution generated by the entrainment of ambient fluid, the concentration of the contaminant diminishes with distance from the discharge location. We shall return to this conclusion in our later analysis of smokestack plumes.

### 3.4 Mixing in Stratified Fluids

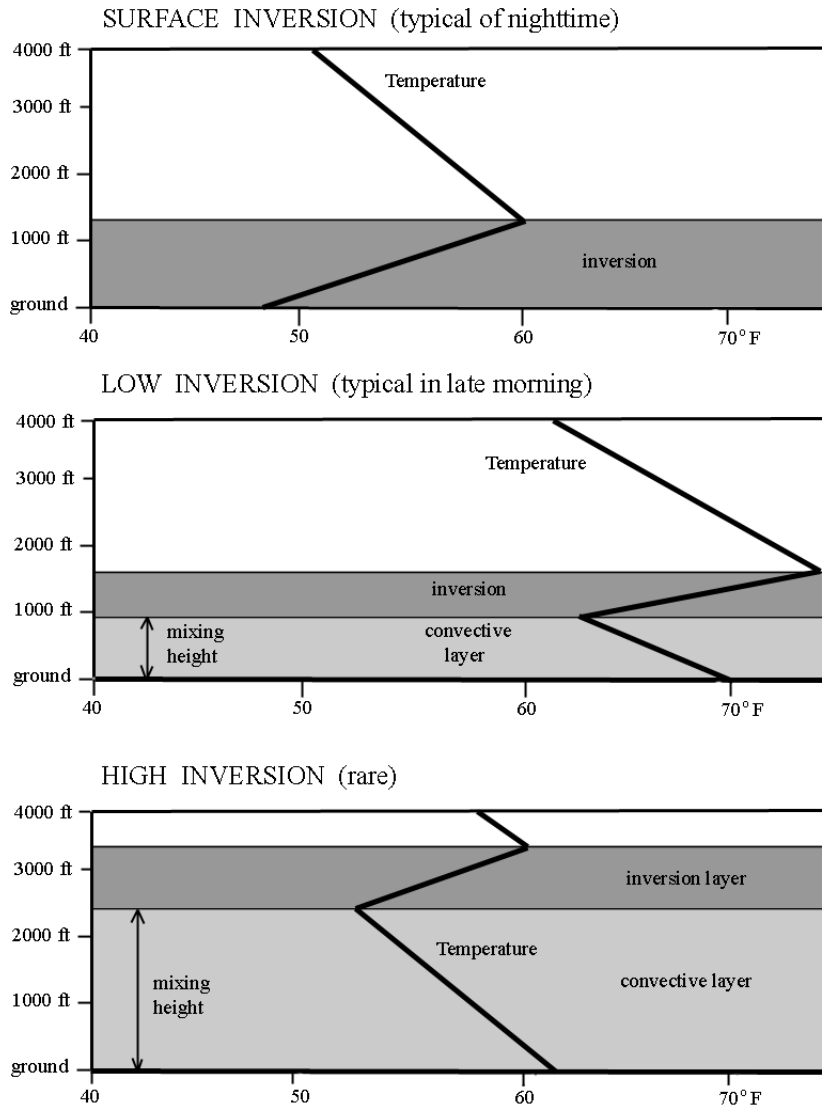
#### *Stratification*

During the summer, the surface water of a lake is typically warmer than the water at depths. Because thermal expansion causes density to decrease with temperature, the surface water is literally floating on top of the colder water below (Figure 3-8). Such situation is called *stratification*. Another example is the lower atmosphere during the night when the ground is colder than the air aloft (Top panel of Figure 3-9).

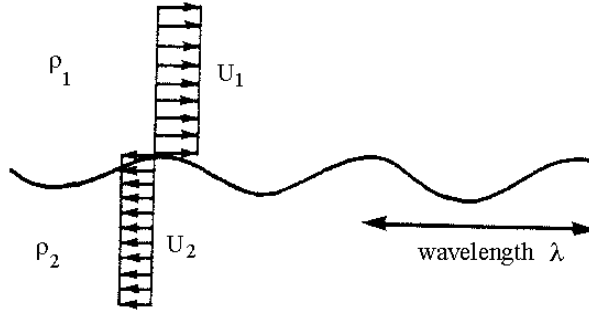
The relation providing density as a function of temperature for a given fluid is its equation of state. Because variations in pressure, temperature and density are relatively modest in most environmental situations, this equation of state can be linearized about a reference state, where the temperature is  $T_0$  and the corresponding density is  $\rho_0$ :



**Figure 3-8.** Thermal stratification in Wellington Reservoir, Western Australia, during summertime (November 1975) and in early winter (February 1976). Note the relatively uniform stratification in the upper half of the water during the summer, caused by solar heating, and the mid-depth sharp stratification in early winter, caused by convective cooling in the upper waters. [From Fischer *et al.*, 1979]



**Figure 3-9.** Examples of atmospheric stratification, showing possible vertical profiles of temperature. In meteorology, a stably stratified layer of air is called an *inversion*. Because vertical mixing of air is severely restricted during an inversion, the presence of a surface inversion aggravates pollution problems. [From Griffin, 1994]



**Figure 3-10.** A two-layer stratification with shear flow. This configuration is conducive to billow development and consequent vertical mixing.

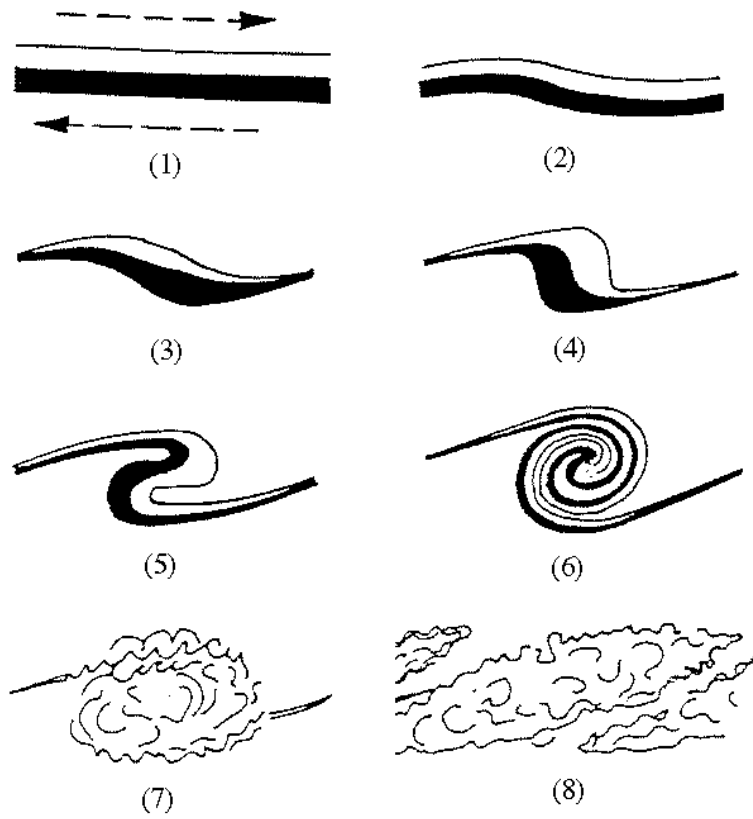
$$\rho = \rho_0 [1 - \alpha (T - T_0)], \quad (3.26)$$

where  $\alpha$  is called the coefficient of thermal expansion (units: per degree Kelvin). For freshwater at most common temperatures and pressures, the thermal expansion coefficient is  $\alpha = 2.57 \times 10^{-4} \text{K}^{-1}$ . For an ideal gas, or a nearly ideal gas such as the air of the atmosphere,  $\alpha$  is equal to  $1/T_0$ , i.e. the inverse of the absolute temperature of the reference state. [A pressure correction to account for the compressibility of the air will be discussed later.] A gravitationally stable stratification (bottom-heavy fluid) occurs whenever the density decreases upward, that is when the temperature increases with height.

Whatever the cause of stratification, it can greatly affect the nature and intensity of mixing. Indeed, mixing in the vertical direction requires overturning, which in this case implies the lifting of heavier fluid from below and lowering of buoyant fluid from above. Both types of motion require work against gravity. From the perspective of energy, some of the kinetic energy of the turbulent motions is diverted from the mixing action and consumed by conversion into potential energy. We therefore conclude that the effect of stratification is most likely a reduction of vertical mixing.

### *Billows*

For the sake of simplicity, let us first consider an extreme type of stratification, namely a two-layer system (Figure 3-10) where a lighter layer of fluid floats over another, heavier layer (such as in a lake in winter – see lower panel of Figure 3-8). Physical principles show that gravity waves can propagate on the interface separating these two layers, not unlike waves propagating on the surface of a pond. If moreover the two layers of the fluid flow at different velocities, i.e. when a shear is present, these waves may grow in time and lead to overturning in the vicinity of the interface. These breaking internal waves, called *billows*, generate mixing over a height a little shorter than their wavelength (Figure 3-11).



**Figure 3-11.** Development of billows at the interface of two fluid layers of different densities and velocities. [From Graf and Mortimer, 1979]

The instability analysis of internal waves along an interface separating two fluid layers of different densities and velocities ( $\rho_1, u_1$  on top, and  $\rho_2, u_2$  below), provides the following criterion: All waves with wavelengths  $\lambda$  satisfying the inequality

$$g\lambda(\rho_2^2 - \rho_1^2) < 2\pi\rho_1\rho_2(u_1 - u_2)^2, \quad (3.27)$$

are unstable. [Here,  $g$  is the gravitational acceleration, equal to  $9.81 \text{ m/s}^2$ .] In other words, all short waves up to a critical wavelength grow in time and turn into billows. This phenomenon is known as the Kelvin-Helmholtz instability. The critical wavelength is

$$\lambda_{\text{crit}} = \frac{2\pi\rho_1\rho_2(u_1 - u_2)^2}{g(\rho_2^2 - \rho_1^2)} \simeq \frac{\pi\rho_0(u_1 - u_2)^2}{g\Delta\rho} = \frac{\pi\Delta u^2}{\alpha g\Delta T}, \quad (3.28)$$

where  $\Delta u = |u_2 - u_1|$  is the velocity difference,  $\Delta\rho = \rho_2 - \rho_1$  is the density difference, assumed to be very small as is usually the case ( $\Delta\rho \ll \rho_1$  and  $\rho_2$ ), and  $\Delta T$  is the temperature difference between the two layers. Laboratory experiments show that billows grow at this critical wavelength,  $\lambda_{\text{crit}}$  (Figure 3-12). Since the amount of overturning is observed to occupy a vertical extent comparable to, but a little shorter than, the billow wavelength, we conclude that vertical mixing occurs over a height of

$$h = C \frac{\Delta u^2}{\alpha g\Delta T}. \quad (3.29)$$

Laboratory measurements reveal that the constant of proportionality  $C$  is about 0.3.

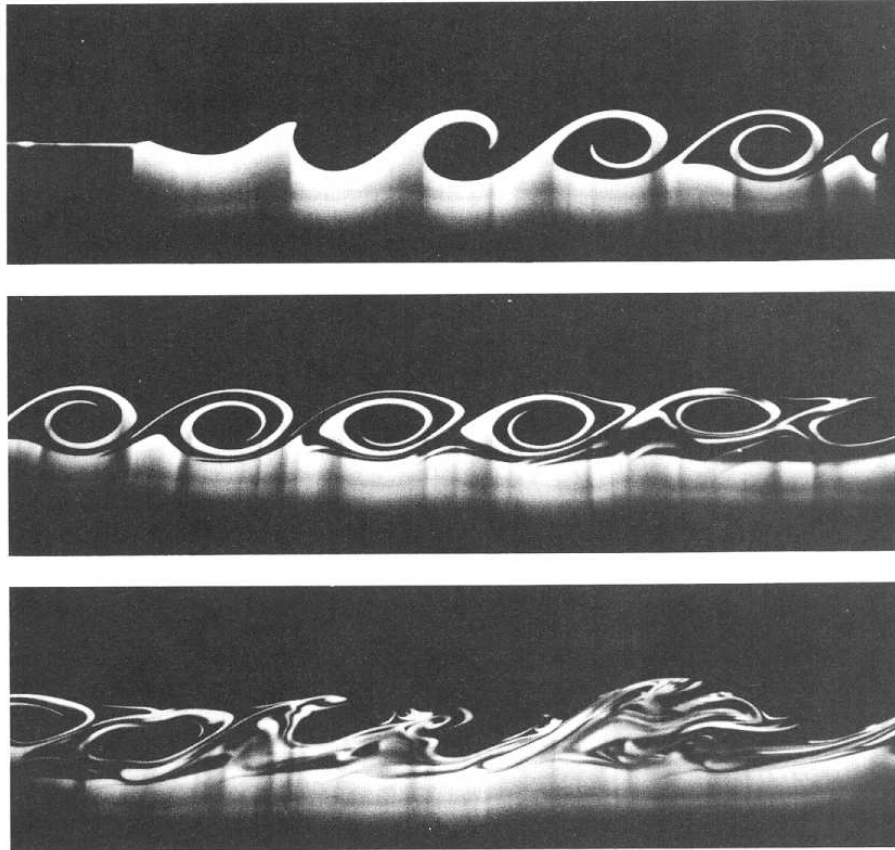
For the turbulence analysis, we take the eddy length and velocity scales to be, respectively,  $d = \Delta u^2/\alpha g\Delta T$  and  $u_* = \Delta u$ . The vigor of vertical overturning leads to rapid homogenization of any contaminant that may be present, in a time interval on the order of  $\tau = d/u_* = \Delta u/\alpha g\Delta T$ . Diffusion in the horizontal proceeds with a diffusion coefficient  $D$  on the order of  $du_* = \Delta u^3/\alpha g\Delta T$ .

#### *Energetics of mixing*

To elucidate the energetics of vertical mixing in the presence of a density difference, consider now the following problem: Initially, there are two layers of fluid of identical depths but different densities and velocities; then, mixing is taking place, leaving a single layer of homogenized fluid, flowing at a uniform velocity (Figure 3-13). Conservation of momentum and heat requires that the new velocity  $u$  be the average of the original velocities  $u_1$  and  $u_2$ , and the new density  $\rho$  be the average of the original densities  $\rho_1$  and  $\rho_2$ :

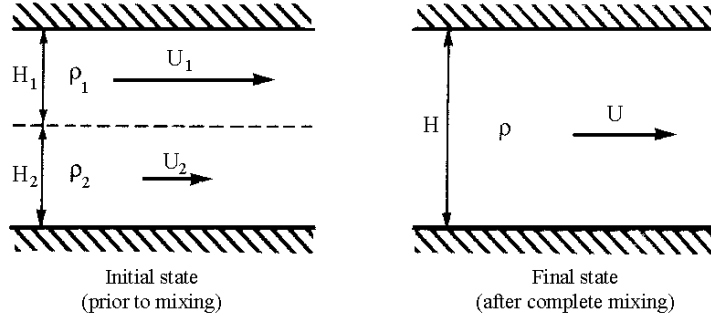
$$u = \frac{1}{2}(u_1 + u_2), \quad \rho = \frac{1}{2}(\rho_1 + \rho_2).$$

Per unit area of the system, the kinetic energy was prior to the mixing event



**Figure 3-12.** Laboratory simulation of the billowing phenomenon. The laboratory conditions are such that distance from the first point of contact between the two fluids (upper left) plays the role of time. Note how the ensuing vertical mixing occurs over a vertical extent close to the billow's initial wavelength. [Photo courtesy of Prof. Greg A. Lawrence, University of British Columbia]





**Figure 3-13.** Mixing of a two-layer stratified fluid with velocity shear. Rising of denser fluid and lowering of lighter fluid require work against buoyancy forces and are possible only if there is a sufficient supply of kinetic energy. Kinetic energy is released from the flow when the velocity shear is reduced.

$$\begin{aligned}
 KE_{\text{initial}} &= \frac{1}{2}\rho_0 u_1^2 \frac{H}{2} + \frac{1}{2}\rho_0 u_2^2 \frac{H}{2} \\
 &= \frac{1}{4}\rho_0(u_1^2 + u_2^2)H
 \end{aligned}$$

and has become afterwards

$$\begin{aligned}
 KE_{\text{final}} &= \frac{1}{2}\rho_0 u^2 H \\
 &= \frac{1}{8}\rho_0(u_1 + u_2)^2 H.
 \end{aligned}$$

There has thus been a drop in kinetic energy:

$$\begin{aligned}
 KE_{\text{drop}} &= KE_{\text{initial}} - KE_{\text{final}} \\
 &= \frac{1}{8}\rho_0(u_1 - u_2)^2 H.
 \end{aligned}$$

Similarly, the potential energy was

$$\begin{aligned}
 PE_{\text{initial}} &= \int_0^{H/2} \rho_2 g z dz + \int_{H/2}^H \rho_1 g z dz \\
 &= \frac{1}{8}\rho_2 g H^2 + \frac{3}{8}\rho_1 g H^2
 \end{aligned}$$

and has become

$$\begin{aligned}
 PE_{\text{final}} &= \int_0^H \rho g z dz \\
 &= \frac{1}{4}(\rho_2 + \rho_1)gH^2,
 \end{aligned}$$

causing a gain of potential energy:

$$\begin{aligned}
 PE_{\text{gain}} &= PE_{\text{final}} - PE_{\text{initial}} \\
 &= \frac{1}{8}(\rho_2 - \rho_1)gH^2.
 \end{aligned}$$

Note how the densities were approximated by the reference density  $\rho_0$  in the expressions of the kinetic energy, where small variations in density do not matter, but not in the expressions of the potential energy, where the small density variations make the whole difference.

Physically, the potential-energy level has been raised because cold fluid has been elevated and light fluid lowered, all against gravity, while kinetic energy has dropped because of dilution. (The square of the average is less than the average of the squares.) Naturally, if the kinetic-energy release exceeds the potential-energy gain, mixing will take place spontaneously. Therefore, the system will homogenize itself if

$$\frac{1}{8}\rho_0(u_1 - u_2)^2H > \frac{1}{8}(\rho_2 - \rho_1)gH^2.$$

In reality, a sizeable fraction of the kinetic energy released by the mixing creates turbulence and, hence, is dissipated by friction. Only the remainder of the energy release serves to increase the potential energy. Laboratory experiments indicate that this remainder is about 30%. Thus, the preceding inequality must be corrected as follows:

$$(0.3)\frac{1}{8}\rho_0(u_1 - u_2)^2H > \frac{1}{8}(\rho_2 - \rho_1)gH^2,$$

i.e., if

$$H < 0.3 \frac{\rho_0(u_1 - u_2)^2}{(\rho_2 - \rho_1)g}. \quad (3.30)$$

Hence, mixing occurs whenever the fluid is sufficiently shallow under given density and velocity differences or, put another way, whenever the velocity shear is sufficiently large or the density difference sufficiently weak.

In the event that inequality (3.26) is not satisfied, there is not enough energy available in the velocity shear for complete mixing. Mixing is then confined to an intermediate height, say  $h$ , that marginally obeys (3.26):

$$\begin{aligned}
 h &= 0.3 \frac{\rho_0(u_1 - u_2)^2}{(\rho_2 - \rho_1)g} \\
 &= 0.3 \frac{\rho_0 \Delta u^2}{g \Delta \rho} = 0.3 \frac{\Delta u^2}{\alpha g \Delta T}.
 \end{aligned} \tag{3.31}$$

Comparing (3.27) to (3.25) and noting that the depth of partial mixing is approximately the vertical extent of wave-induced overturning, we conclude that billow formation is governed by a balance between release of kinetic energy from the flow and consumption of potential energy by mixing against gravity.

In practice, it is helpful to codify the preceding considerations by introducing a dimensionless number, called the Richardson number:

$$Ri = \frac{\alpha g h \Delta T}{\Delta u^2}, \tag{3.32}$$

where  $h$  is the fluid depth under consideration (either total depth or depth of a turbulent layer, whichever the context dictates is the most appropriate),  $\Delta T$  is a measure of the vertical temperature difference across the depth  $h$ , and  $\Delta u$  is a measure of the velocity shear across that same layer. The physics tell us that:

- If  $Ri > 0.3$ , the flow is stable, no mixing occurs;
- If  $Ri < 0.3$ , the flow is unstable, mixing occurs, and  $h$  grows until  $Ri$  reaches 0.3.

The time scale over which mixing occurs, should it occur, is usually quite fast:

$$\begin{aligned}
 \text{mixing time} &\simeq \frac{\text{depth of mixing}}{\text{velocity of overturn}} \\
 &= \frac{\Delta u^2 / \alpha g \Delta T}{\Delta u} = \frac{\Delta u}{\alpha g \Delta T}.
 \end{aligned} \tag{3.33}$$