ENVIRONMENTAL FLUID MECHANICS

Turbulence

Hand sketch of turbulent flow by Leonardo da Vinci (1452-1519)

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Homogeneous and Isotropic Turbulence

Turbulent flow field seen as a population of eddies of various sizes and orbital speeds
Distribution of turbulent vortices in homogeneous & isotropic turbulence

Assumption has been made that all turbulent eddies of same diameter $d$ share the same orbital velocity $\tilde{u}$.

There exists therefore a function $\tilde{u}(d)$.

Concept of the energy cascade

energy supplied by external forces

flow of energy

energy removed by viscous forces
Kolmogorov Eddy Cascade Theory:

Essentially a dimensional argument

The driver of turbulence is the energy \( \varepsilon \) passing through the cascade, which is energy per time and per mass (units: \( \text{W/kg} = \text{m}^2/\text{s}^3 \)).

The only way to make the velocity \( \bar{u}(d) \) (units \( \text{m/s} \)) be a function of \( d \) (units \( \text{m} \)) is by the following power law:

\[
\bar{u}(d) = A \left( \varepsilon d \right)^{1/3}
\]

with \( A \) being a universal dimensionless constant.

Turbulent energy spectrum for homogeneous and isotropic turbulence.

The \(-5/3\) power decay agrees with the Kolmogorov theory.
Shear Turbulence

Typical velocity distributions across realistic channels. Velocities are largest away from the bottom and sides.

Turbulent flow along a wall

Side view
Wall is the bottom boundary. Flow is from left to right.

Top view
Hydrogen bubbles are generated at line on left, and flow proceeds to the right.
Hairpin vortex in shear turbulence in proximity of a wall

Evidence of logarithmic velocity profile
Turbulent flow along a rough boundary

Velocity profile is still logarithmic but no longer dependent on the fluid’s viscosity.
Instead, a new parameter, the roughness height $z_o$, enters the expression.

**Roughness heights**

<table>
<thead>
<tr>
<th>Type of surface</th>
<th>$z_o$ (m)</th>
<th>Reference</th>
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</thead>
<tbody>
<tr>
<td><strong>Atmosphere</strong></td>
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<tr>
<td>Sand, snow or sea</td>
<td>0.01m $z_o$</td>
<td>Chamberlain (1983)</td>
</tr>
<tr>
<td>Active sand sheet</td>
<td>0.0064</td>
<td>Draper et al. (2001)</td>
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<tr>
<td>Sand sheet</td>
<td>0.0002-0.005</td>
<td>Draper et al. (2001)</td>
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<tr>
<td>Gravel lag</td>
<td>0.002</td>
<td>Draper et al. (2001)</td>
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<tr>
<td>Bare soil</td>
<td>0.03-0.1</td>
<td>Garriott (1992)</td>
</tr>
<tr>
<td>Flat open country</td>
<td>0.02-0.06</td>
<td>Davenport (1965)</td>
</tr>
<tr>
<td>Grass - sparse</td>
<td>0.0012</td>
<td>Clarke et al. (1971)</td>
</tr>
<tr>
<td>Grass - thick</td>
<td>0.026</td>
<td>Su et al. (2001)</td>
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<td>Grass - thin</td>
<td>0.05</td>
<td>Sutton (1993)</td>
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<td>Wheat field</td>
<td>0.015</td>
<td>Garriott (1997)</td>
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<tr>
<td>Corn field</td>
<td>0.04</td>
<td>Kung (1961)</td>
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<tr>
<td>Vines - along rows</td>
<td>0.025</td>
<td>Hicks (1973)</td>
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<td>Vines - across rows</td>
<td>0.12</td>
<td>Hicks (1973)</td>
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<tr>
<td>Vegetation</td>
<td>0.2</td>
<td>Fichill and McVehil (1970)</td>
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<tr>
<td>Swamp</td>
<td>0.4</td>
<td>Garriott (1980)</td>
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<tr>
<td>Scrub</td>
<td>0.049</td>
<td>Su et al. (2001)</td>
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<tr>
<td>Trees</td>
<td>0.4</td>
<td>Fichill and McVehil (1970)</td>
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<tr>
<td>Trees - sparse</td>
<td>0.1 of tree height</td>
<td>Gupta and Seguin (1978)</td>
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<tr>
<td>Forest - temperate</td>
<td>0.26-0.92</td>
<td>Hicks et al. (1975),</td>
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<td></td>
<td></td>
<td>Fohn et al. (1975),</td>
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<td></td>
<td></td>
<td>Jarvis et al. (1976),</td>
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<td></td>
<td></td>
<td>Thomson and Flander (1975),</td>
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<td></td>
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<td>Shuttlesworth (1969)</td>
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<td>Forest - tropical</td>
<td>2.2-4.8</td>
<td>Davenport (1965)</td>
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<td>Residential neighborhood</td>
<td>1.0-3.0</td>
<td>Davenport (1965)</td>
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<td>City with high rises</td>
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<tr>
<td>Hilly terrain</td>
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<td><strong>Winter</strong></td>
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<td>Concrete channel</td>
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<td>Mud</td>
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<td>Course gravel</td>
<td>2.0 $z_o$</td>
<td>Kasprians (1974)</td>
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<tr>
<td>Rocks</td>
<td>0.05</td>
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Drag Coefficient

The drag coefficient $C_D$ is defined as the coefficient of proportionality between bottom stress $\tau_b$ and square of velocity:

$$\tau_b = C_D \rho \overline{u^2}$$

in which $\overline{u}$ is the average velocity.

We can calculate the drag coefficient for the logarithmic profile along a rough wall if we also know the height $h$ of the fluid:

$$u(z) = \frac{u_*}{\kappa} \ln \frac{z}{z_0} \quad 0 < z < h \quad \Rightarrow \quad \overline{u} = \frac{1}{h} \int_0^h u(z) \, dz = \frac{u_*}{\kappa} \left[ \ln \frac{h}{z_0} - 1 \right]$$

from which follows a relation between the turbulent velocity and average velocity:

$$u_* = \frac{k \overline{u}}{\ln(\overline{h}/z_0) - 1}$$

The drag coefficient follows:

$$C_D = \frac{\tau_b}{\rho \overline{u^2}} = \frac{\rho u_*^2}{\rho \overline{u^2}} \left( \frac{\overline{u}}{u_*} \right)^2 = \left( \frac{k}{\ln(\overline{h}/z_0) - 1} \right)^2$$

For $\kappa = 0.41$ and $h/z_0$ in the range 50 – 1000, $C_D$ varies between 0.005 and 0.020.
Eddy Viscosity & Mixing Length

In an attempt to apply the formulation of wall turbulence to more general turbulent flows, the concepts of eddy viscosity and mixing length are introduced.

The eddy viscosity is the apparent viscosity of the turbulent fluid:

\[ \tau_e = \rho \nu T \frac{\partial u}{\partial z} \]

In analogy with molecular viscosity \( \tau = \rho \nu (\partial u / \partial z) \), we write:

\[ \tau_e = \rho \nu_T (\partial u / \partial z) \]

in which \( \nu_T \) is called the eddy viscosity.

For wall turbulence, we have:

\[ u(z) = \frac{u_*}{k} \ln \frac{z}{\theta_e} \rightarrow \frac{\partial u}{\partial z} = \frac{u_*}{k} \rightarrow \nu_T = \frac{\tau_e}{\rho (\partial u / \partial z)} = \kappa u_z = (\kappa z) u_* = \ell_m u_* \]

The eddy viscosity can thus be interpreted as the product of a mixing length \( \ell_m \) with the turbulent velocity.

For wall turbulence, the mixing length is \( \ell_m = \kappa z \) and the turbulent velocity is obtained from \( u_*^2 = \gamma / \rho \).

The question is:

What should the mixing length and turbulent velocity be in more general turbulent flows?

Thought: Turbulence is the result of instabilities, and instabilities are caused by shear in the flow.

So, connect with the flow shear:

\[ \tau_e = \rho \nu_T \frac{\partial u}{\partial z} \rightarrow \rho u_*^2 = \rho (\ell_m u_* \frac{\partial u}{\partial z}) \rightarrow u_* = \ell_m \frac{\partial u}{\partial z} \]

\[ \nu_T = \ell_m u_* = \ell_m \frac{\partial u}{\partial z} \]

Generalization to more general, 3D turbulent flows:

1) Define strain tensor as

\[ S_{ij} = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \]

and overall strain \( S \) from

\[ S^2 = 2 \sum_{i=1}^{3} \sum_{j=1}^{3} S_{ij}^2 \]

This strain \( S \) can be viewed as the generalization of the shear \( \partial u / \partial z \) and thus:

\[ \nu_T = \ell_m^2 S \]

2) Better: Define the rate-of-rotation tensor

\[ \Omega_{ij} = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i} \right) \]

and overall shear \( \Omega \) from

\[ \Omega^2 = 2 \sum_{i=1}^{3} \sum_{j=1}^{3} \Omega_{ij}^2 \]

from which follows the alternative formulation

\[ \nu_T = \ell_m^2 \Omega \]
And what about the mixing length?

For wall turbulence, \( \ell_m = \kappa z \), which suggests a geometric relation.

One possibility is: \( \ell_m = \kappa \) times distance to closest wall.

For channel flow (with two walls, one on each side, say at \( y = 0 \) and \( y = W \)), one often takes:

\[
\ell_m = \kappa \frac{y (W - y)}{W}
\]