ENVIRONMENTAL FLUID MECHANICS

Instabilities

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Kelvin-Helmholtz instability

(Thorpe, 1971)
With downstream distance replacing time

faster & lighter

slower & denser

(Greg Lawrence, 1991)
It is believed that the geometric meander pattern so common on antique artifacts was inspired by the Kelvin-Helmholtz instability.
Kelvin-Helmholtz instability theory

Separate variables into basic flow + perturbation

Substitute in governing equations and linearize by assuming that perturbation remains small over time

Seek a solution of the type

Equations reduce to a simple problem in $z$: Solution is:

If we introduce the displacement $a(x, t)$ of the interface and ascribe to it a similar wave form:

then matching the vertical velocity $u$ with this interfacial displacement ($u = i\theta j/dt$ $\rightarrow$ $u' = \partial u/\partial z + i\theta j$) gives:

Upper layer: $W = i((k\theta j - \omega)A e^{-kz}$

Lower layer: $W = i((k\theta j - \omega)A e^{+kz}$

Upper layer: $P = -\frac{\eta_0(\theta j_0 - \omega)^2}{k} A e^{-kz}$

Lower layer: $P = \frac{\eta_0(\theta j_0 - \omega)^2}{k} A e^{+kz}$
Matching of pressure at the interface between the two layers yields

$$\frac{\Delta \rho}{\rho_0} - ik(\Delta \rho) = (k\Delta \rho_1 - \omega)^2 + (k\Delta \rho_2 - \omega)^2 \Delta \rho = \rho_2 - \rho_1$$

$$\omega = \frac{1}{2} \left[k(\Delta \rho_1 + \Delta \rho_2) \pm \sqrt{k^2(\Delta \rho_1 - \Delta \rho_2)^2 - k^2(\Delta \rho_1 - \Delta \rho_2)^2} \right]$$

The frequency $\omega$ is real-valued as long as

$$2g \frac{\Delta \rho}{\rho_0} > \epsilon(\Delta \rho)^2 \Delta \rho = \rho_2 - \rho_1 \Delta \rho = |\Delta \rho_1 - \Delta \rho_2|$$

But, should this not be the case, then the frequency $\omega$ is complex, with $\pm$ in front of the imaginary part. One of these two roots has a positive imaginary part and corresponds to a wave growing exponentially over time. There is instability in this case.

Put in terms of the wavelength $\lambda$, instability occurs when:

$$\lambda < \frac{\tau \eta \Delta \rho^2}{\rho \Delta \rho}$$

$$\lambda_{cr} = \frac{\tau \eta \Delta \rho^2}{\rho \Delta \rho} = \frac{\pi \Delta \rho^2}{\rho \Delta \rho}$$

There is a critical wavelength that separates stable from unstable waves.

So, we have

$$\lambda_{cr} = \frac{\pi \Delta \rho^2}{\rho \Delta \rho}$$

We do not know which wavelength among the unstable ones ($\lambda < \lambda_{cr}$) will dominate the finite-amplitude regime, but using $\lambda_{cr}$ as its length scale seems appropriate, especially since merging of several waves takes place during the rolling process. So, while $\lambda$ begins below $\lambda_{cr}$, it may not be far from $\lambda_{cr}$ by the time the successive mergers have taken place.

We further notice from laboratory observations and numerical simulations that the vertical extent of the elongated waves and the resulting thickness of the mixed turbulent zone is approximately one third of the wavelength.

Thus, the height $h$ over which mixing ensues is about

$$h = \frac{\Delta \rho}{3} \frac{\lambda_{cr}}{3} \frac{\Delta \rho}{3 \rho \Delta \rho} - \frac{\Delta \rho}{3 \rho \Delta \rho}$$

leading to

$$\frac{\rho_1 \Delta \rho}{\Delta \rho^2} - \frac{\rho_2 \Delta \rho}{\Delta \rho^2} - 1$$

We recognize here the Richardson number.
Since the theory leads to a Richardson number expression, we conclude that the physics at work include an exchange between kinetic and potential energy.

It is clear that mixing implies the raising of denser fluid and the lowering of lighter fluid, both having to be accomplished by overcoming buoyancy forces (raising of the level of the center of gravity of the system). This potential energy is consumed in the process.

The source of this new potential energy is partial consumption of kinetic energy in the flow.

Thus, instability and mixing are in essence driven by a spontaneous consumption of kinetic energy and its conversion into added potential energy.

Note that the conversion is only partial (efficiency < 100%) because some of the kinetic energy evidently goes into turbulent motion and ultimate mechanical dissipation into heat, leaving only a fraction going to potential energy.

The outcome of the situation is:

If we conjecture that mixing proceeds only as much as it needs to stabilize the system and no more, then the final continuous density stratification and velocity shear in the mixed zone may be considered as marginally stable.

The question becomes: What is the marginally stable state of a continuous density stratification in the presence of velocity shear?
Instability of a Stratified Shear Flow

We now consider the continuous extension of the previous two-layer analysis. There is continuous velocity shear and continuous density stratification.

\[ \pi(z) \text{ with } \frac{d\pi}{dz} \neq 0 \] (any sign)

\[ \pi(z) \text{ with } N^2 = -\frac{g}{\rho_0} \frac{d\pi}{dz} > 0 \] (lighter on top, denser below)

To make a long story short:
- Restrict attention to 2D vertical plane \((x,z)\)
- Write governing equations (volume conservation + 2 momentum equations + energy equation)
- Split flow and density variations into basic state + perturbation
- Linearize equations by assuming weak perturbations
- Seek solution of the type:

\[
\begin{align*}
\psi(x,z,t) &= \psi(z) e^{i(kx - \omega t)} \\
\psi(z) &= \Psi(z) e^{i(kz - \omega t)}
\end{align*}
\]

with \(k\) real positive and \(\omega\) possibly complex

\[ \omega = \omega_r + i \omega_i \]

Algebra yields a single equation for the wave amplitude \(\Psi(z)\):

\[
(\pi - c) \left( \frac{d^2\Psi}{dz^2} - k^2 \Psi \right) + \left( \frac{N^2}{\pi - c} \right) \frac{d^2\Psi}{dz^2} = 0
\]

in which \(c = \frac{\omega_r}{k} = c_r + i c_i\) is the (possibly complex) wave speed.

For impermeable boundary conditions at some bottom \((z = 0)\) and top \((z = H)\), the conditions on the wave amplitude are

\[ \Psi(0) = \Psi(H) = 0 \]

This problem is very complicated and has only been solved analytically in a few simple cases.

Here, we will derive a general criterion that circumvents the need to find the analytical solution to the problem.
The trick is to introduce the new function
\[ \phi(z) = \frac{\psi(z)}{\sqrt{c - c}} \]

which obeys the slightly different equation
\[
\frac{d}{dz} \left[ (c - c) \frac{d\phi}{dz} \right] - \left[ k^2 (c - c) + \frac{1}{2} \frac{d^2}{dz^2} \psi \phi^2 - \frac{4N^2}{4(c - c)} \right] \phi = 0
\]
accompanied by unchanged boundary conditions
\[ \phi(0) = \phi(H) = 0 \]

After multiplication of the equation by the complex conjugate \( \psi^* \) of \( \psi \) and integrating over the height of the domain, we obtain after some integration by parts:
\[
\int_0^H (c - c) \left( \frac{d\phi}{dz} \right)^2 + k^2 \left| \psi \right|^2 dz + \frac{1}{2} \int_0^H \psi \frac{d^2}{dz^2} \left| \psi \right|^2 dz + \int_0^H (\psi / dz)^2 - 4N^2 \left( \psi / (c - c) \right) \left| \psi \right|^2 dz = 0
\]

Of the three terms, the middle one is always real, but the first and last may be complex since the wave speed \( c \) may include an imaginary component (\( c_i = c / k \)) if the frequency \( \omega \) happens to be complex.

Extracting the imaginary part of the previous equation yields:
\[
c_i \int_0^H \left( \frac{d\phi}{dz} \right)^2 + k^2 \left| \psi \right|^2 \right) dz = \frac{c_i}{4} \int_0^H \left( \frac{d\phi}{dz} \right)^2 dz - 4N^2 \frac{\left| \psi \right|^2}{(c - c)} dz
\]

The discussion then proceeds as follows. Either \( c_i \) is zero or it is not.

If \( c_i = 0 \), the frequency \( \omega \) is real, and the wave perturbation does not grow. It is stable.

Hence, instability can only occur if \( c_i \neq 0 \), in which case we may divide both sides of the previous equation by \( c_i \), which then requires that the two integrals be equal to each other.
Put another way, instability requires at a minimum that

\[
\int_0^1 \left( \frac{dU}{dz} + k^2 \rho \right) \, dz = \frac{1}{4} \int \left( \frac{d\phi}{dz} \right)^2 - 4N^2 \left( \rho - \rho_0 \right) \, dz
\]

Since the integral on the left is clearly always positive, the one on the right equal to it must also be positive. This in turn requires that its integrand be at least positive in some portion of the domain.

The inequality

\[
\left( \frac{d\phi}{dz} \right)^2 > 4N^2
\]

is a required condition before instability can occur.

If both velocity and density vary linearly with height across the domain, then

\[
\frac{dU}{dz} = \frac{\Delta U}{H}, \quad \frac{d\phi}{dz} = -\frac{\Delta \rho}{H} \Rightarrow N^2 = \frac{g \Delta \rho}{\rho_0 H} \Rightarrow Ri = \frac{gH \Delta \rho}{\rho_0 AU^2} < \frac{1}{4}
\]

Another change of variable, this time

\[
a(z) = \frac{\Psi(z)}{\pi - c}
\]

leads to the so-called Howard’s semi-circle theorem, which places bounds on the real and imaginary parts of the wave speed.

(No demonstration here. See Chapter 5 for the analysis.)
Computer simulation by method of contour dynamics

Extreme case of no stratification:

\[ N^2 = 0 \] and all that is required is

\[ \frac{d^2 \pi}{dz^2} = 0 \]

somewhere in the domain.