Ship Waves

Material drawn from "Waves in Fluids" by James Lighthill
Section 3.10

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Ship waves are deep-water gravity waves because their wavelengths $\lambda$ are on the order of the ship’s length $L$, which is typically much shorter than the water depth $H$:

$$\lambda \sim L < \ll H$$

Their dispersion relation is

$$\omega = \sqrt{gk} = \sqrt{\frac{2\pi g}{\lambda}}$$

from which follows their phase speed

$$c = \frac{\omega}{k} = \frac{g}{k} = \frac{g\lambda}{2\pi}$$

and their group velocity

$$c_g = \frac{d\omega}{dk} = \frac{1}{2} \frac{g}{k} = \frac{1}{2} \frac{g\lambda}{2\pi}$$

Stationarity with respect to the ship requires the waves to have a phase speed in the direction of the ship that equals the speed of the ship.

$$ct = Ut \cos \theta$$

$$c = U \cos \theta$$
But since phase speed varies with wavelength, there are many waves that meet the criterion.

\[ c = \frac{g\lambda}{2\pi} = U \cos \theta \]

smaller angle \( \theta \) \( \rightarrow \) longer \( \lambda \)

Locus of end points of the wave phase travel \( ct \) is a circle drawn with diameter equal to the distance \( Ut \) traveled by the ship.

Next but: The energy in the waves, and therefore the location where we observe the wave packets, travels not at the phase speed but at the group velocity.

\[ c_g = \frac{c}{2} \]

Since the group velocity is only half of the phase speed for deep-water gravity waves, the energy is located only within a circle of half the size.
Now consider many different earlier times $t$.

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Determination of the wake region angle

Thus, the half angle of the wake is $19.5^\circ$ and the full angle is $39^\circ$. 

\[
\sin \beta = \frac{Ut/4}{Ut/2 + Ut/4} = \frac{1}{3/4} = \frac{1}{3}
\]

\[
\Rightarrow \quad \beta = \sin^{-1}\left(\frac{1}{3}\right) = 19.5^\circ
\]
Now consider the angle of the wave crests with respect to the edge of the wake.

\[ 2\theta + 90^\circ + \beta = 180^\circ \]

\[ \rightarrow \theta = 35.3^\circ \]

and \[ 90^\circ - \theta = 54.7^\circ . \]

Thus, the angle made by the wave crests with respect to the ship’s path is 55°.
So, now when you happen to be by the water with your beloved, and the scene is most romantic…

… and the sun sets.

All you can now think about is: wake at 35° and waves at 55°, because Professor Cushman-Roisin taught you so.

One can also calculate the phase speed of the waves forming the wake:

\[ c = U \cos \theta \]

\[ = U \cos(35.26°) \]

\[ = \frac{2}{\sqrt{3}} U = 0.816 U \]

and their wavelength:

\[ c = \sqrt{\frac{gA}{2\pi}} \Rightarrow \lambda = \frac{2\pi}{g} \frac{c^2}{g} = \frac{2U^2}{3g} = \frac{4\pi U^2}{3g} = 4.19 \frac{U^2}{g} \]
A ship creates a wave crest at its bow (front) and a trough at its stern (rear), thus generating waves with wavelengths $\lambda$ comparable to its length $L$.

With the requirement that $\lambda = 4.19 \frac{U^2}{g}$, this is possible only if the ship’s length is on the order of $\frac{U^2}{g}$. What happens otherwise?

**Outlying cases**

1. Very long ship: $L \gg \frac{U^2}{g}$  
   The wave pattern is dominated by waves around the maximum wavelength $2\pi \frac{U^2}{g}$ moving forwards behind the ship at small angles to the ship’s path.

2. Very short ship:  
   (ex. speedboats)  
   $L \ll \frac{U^2}{g}$  
   The wave pattern is dominated by waves much shorter than the maximum wavelength $2\pi \frac{U^2}{g}$ and these waves propagate at large angles to the ship’s path.

The preceding discussion points to the importance of the ratio

$$Fr^2 = \frac{U^2}{gL} \Rightarrow Fr = \frac{U}{\sqrt{gL}}$$

in ship design. This is called the **Froude Number** in honor of William Froude (1810-1879).

The Froude number is essential in scaling down ship hulls and speeds for laboratory testing at the design stage.
Question

What is the advantage of a bulb in the front of the hull?

Wave drag of a ship

Dimensional analysis suggests $P_w \propto \rho U^3 L^2$

Convention leads to the definition of a wave drag: $\frac{P_w}{U} = D_w \rho U^2 L^2$

Note that the ship’s resistance to motion also includes a skin friction drag.
The End