Chapter 8

Turbulence

SUMMARY: Almost all environmental fluid flows, be they of air or water, are turbulent. Unfortunately, the highly intermittent and irregular character of turbulence defies analysis, and there does not yet exist a unifying theory of turbulence, not even one for its statistical properties. So, the approach in this chapter is necessarily much more empirical and heuristic. Each section considers turbulence in on of its most common manifestations in natural fluid flows: homogeneous turbulence, shear turbulence, and turbulence in stratified fluids. It then explores several classical methods to model these types of turbulence.

8.1 Homogeneous and Isotropic Turbulence

At a very basic level, a turbulence flow can be interpreted as a population of many eddies (vortices), of different sizes and strengths, embedded in on another and forever changing, giving a random appearance to the flow (Figure 8.1). Two variables then play a fundamental role: d, the characteristic diameter of the eddies, and $\mathring{u}$, their characteristic orbital velocity. Since the turbulent flow consist in many eddies, of varying sizes and speeds, $\mathring{u}$ and d do not assume each of a single value but vary within a certain range. In stationary, homogeneous and isotropic turbulence, that is, a turbulent flow that statistically appears unchanging in time, uniform in space and without preferential direction, all eddies of a given size (same d) behave more or less in the same way and can be thought of sharing the same characteristic velocity $\mathring{u}$. In other words, we make the assumption that $\mathring{u}$ is a function of d (Figure 8.2).

8.1.1 Energy cascade

In the view of Kolmogorov (1941), turbulent motions span a wide range of scales ranging from a macroscale at which the energy is supplied, to a microscale at which energy is dissipated by viscosity. The interaction among the eddies of various scales

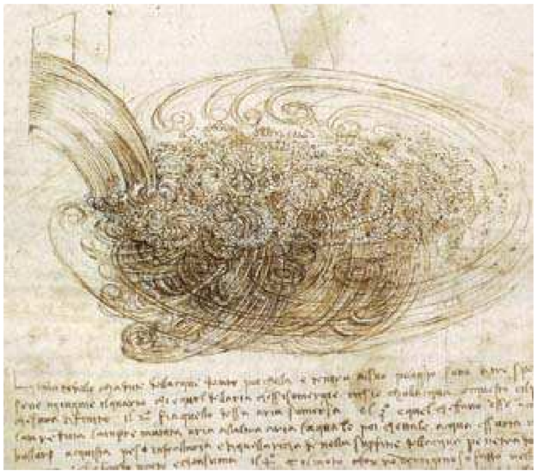

Figure 8.1: Drawing of a turbulent flow by Leonardo da Vinci (1452–1519), who recognized that turbulence involves a multitude of eddies at various scales.

passes energy sequentially from the larger eddies gradually to the smaller ones. This process is known as the *turbulent energy cascade* (Figure 8.3).

If the state of turbulence is statistically steady (statistically unchanging turbulence intensity), then the rate of energy transfer from one scale to the next must be the same for all scales, so that no group of eddies sharing the same scale sees its total energy level increase or decrease over time. It follows that the rate at which energy is supplied at the largest possible scale ($d_{\max}$) is equal to that dissipated at the shortest scale ($d_{\min}$). Let us denote by ϵ this rate of energy supply/dissipation, per unit mass of fluid:

$$\begin{aligned}
\epsilon &= \text{energy supplied to fluid per unit mass and time} \\
&= \text{energy cascading from scale to scale, per unit mass and time} \\
&= \text{energy dissipated by viscosity, per unit mass and time.}
\end{aligned}$$

The dimensions of ϵ are:

$$[\epsilon] \;=\; \frac{ML^2T^{-2}}{MT} \;=\; L^2T^{-3}. \tag{8.1}$$

With Kolmogorov, we further assume that the characteristics of the turbulent eddies of scale d depend solely on d itself and on the energy cascade rate ϵ. This is to mean that the eddies know how big they are, at which rate energy is supplied to them and at which rate they must supply it to the next smaller eddies in the cascade. Mathematically, $\mathring{u}$ depends only on d and ϵ. Since $[\mathring{u}] = LT^{-1}$, $[d] = L$ and $[\epsilon] = L^2T^{-3}$, the only dimensionally acceptable possibility is:

$$\mathring{u}(d) \;=\; A(\epsilon d)^{1/3}, \tag{8.2}$$

in which A is a dimensionless constant on the order of unity.

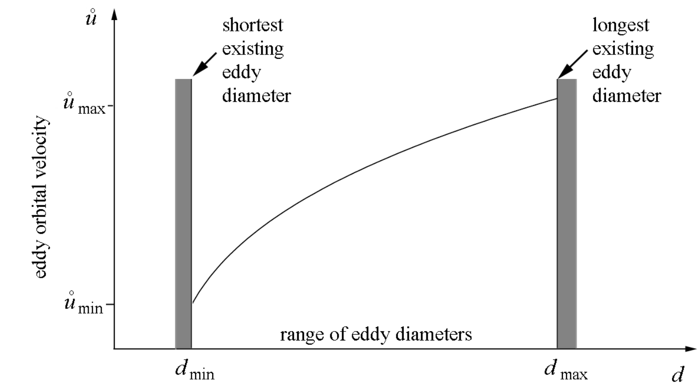

Figure 8.2: Eddy orbital velocity versus eddy length scale in homogeneous turbulence. The largest eddies spin the fastest.

Thus, the larger ϵ, the larger $\mathring{u}$. This makes sense, for a greater energy supply to the system generates stronger eddies. Equation (8.2) further tells that the smaller d, the weaker $\mathring{u}$. This could not have been anticipated and must be accepted as a result of the theory. The implication is that the smallest eddies have the lowest speeds, while the largest ones have the highest speeds and thus contain the bulk of the kinetic energy.

8.1.2 Largest and shortest length scales

Typically, the largest possible eddies in the turbulent flow are those that extend across the entire system, from boundary to opposite boundary, and therefore

$$d_{\text{max}} = L, \tag{8.3}$$

where L is the geometrical dimension of the system (such as the width of the domain or the cubic root of its volume). In environmental flows, there is typically a noticeable scale disparity between a relatively short vertical extent (depth, height) and a comparatively long horizontal extent (distance, length) of the system. Examples are:

rivers :	depth	<<	width <<	length
atmosphere :	height	<<	physically relevant horizontal distances.	

In such situations, we must clearly distinguish eddies that rotate in the vertical plane (about a horizontal axis) from those that rotate horizontally (about a vertical axis). In rivers, we may furthermore distinguish transverse eddies from longitudinal eddies.

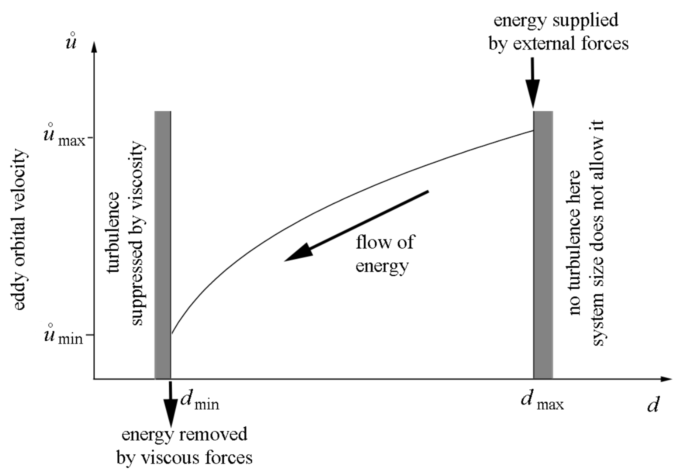

Figure 8.3: The turbulent energy cascade. According to this theory, the energy fed by external forces excites the largest possible eddies and is gradually passed to ever smaller eddies, all the way to a minimum scale where this energy is ultimately dissipated by viscosity.

The shortest eddy scale is set by viscosity, because the shorter the eddy scale, the stronger the velocity shear and the more important the effect of viscosity. Consequently, the shortest eddy scale can be defined as the length scale at which viscosity becomes dominant. Viscosity, denoted by ν, has for dimensions[1]:

$$[\nu] \;=\; L^2 T^{-1}.$$

If we assume that $d_{\min}$ depends only on ϵ, the rate at which energy is supplied to that scale, and on ν, because those eddies sense viscosity, then the only dimensionally acceptable relation is:

$$d_{\min} \;\sim\; \nu^{3/4}\epsilon^{-1/4}. \tag{8.4}$$

Therefore, $d_{\min}$ depends on the energy level of the turbulence. The stronger the turbulence (the bigger ϵ), the shorter the minimum length scale at which it is capable of stirring. The quantity $\nu^{3/4}\epsilon^{-1/4}$, called the *Kolmogorov scale*, is typically on the order of a few millimeters or shorter.

The span of length scales in a turbulent flow is related to its Reynolds number. Indeed, in terms of the largest velocity scale, which is the orbital velocity of the largest eddies, $U = \mathring{u}(d_{\max}) = A(\epsilon L)^{1/3}$, the energy supply/dissipation rate is

$$\epsilon \;=\; \frac{U^3}{A^3 L} \;\sim\; \frac{U^3}{L}\;, \tag{8.5}$$

[1]Values for ambient air and water are: $\nu_{\text{air}} = 1.51\times10^{-5}$ m^2/s and $\nu_{\text{water}} = 1.01\times10^{-6}$ m^2/s.

Figure 8.4: A real water cascade, showing the tumbling down of water from the highest to the lowest point. In analogy, energy of a turbulent flow is tumbling down from the largest to the shortest eddy scale. [Photo ©Ian Adams]

and the length scale ratio can be expressed as

$$\frac{L}{d_{\min}} \sim \frac{L}{\nu^{3/4}\epsilon^{-1/4}} \sim \frac{LU^{3/4}}{\nu^{3/4}L^{1/4}} \sim Re^{3/4}, \tag{8.6}$$

where $Re = UL/\nu$ is the Reynolds number of the flow. As we could have expected, a flow with a higher Reynolds number contains a broader range of eddies.

Example 8.1 Atmospheric turbulence

Consider the atmospheric boundary layer, spanning a height of about 1000 m above the ground. If the typical wind speed is 10 m/s, then the Reynolds number can be estimated to be

$$Re = \frac{UL}{\nu} = \frac{(10 \text{ m/s})(1000 \text{ m})}{(1.51 \times 10^{-5} \text{ m}^2/\text{s})} = 6.6 \times 10^8.$$

which yields $Re^{3/4} = 4.1 \times 10^6$ and

$$d_{\min} \sim \frac{L}{Re^{3/4}} = 2.4 \times 10^{-4} \text{ m}$$

or about 0.24 mm.

The energy supply/dissipation rate is estimated to be

$$\epsilon \sim \frac{U^3}{L} = \frac{(10 \text{ m/s})^3}{(1000 \text{ m})} = 1 \text{ m}^2/\text{s}^3.$$

or 1 Joule per kilogram of air and per second.

8.1.3 Energy spectrum

In turbulence theory, it is customary to consider the so-called *power spectrum*, which is the distribution of kinetic energy per mass across the various length scales. For this, we need to define a wavenumber. Because velocity reverses across the diameter of an eddy, the eddy diameter should properly be considered as half of the wavelength:

$$k \;=\; \frac{2\pi}{\text{wavelength}} \;=\; \frac{\pi}{d} \; . \tag{8.7}$$

The extremal values are $k_{\min} = \pi/L$ and $k_{\max} \sim \epsilon^{1/4}\nu^{-3/4}$.

The kinetic energy E per mass of fluid has dimensions of $\mathrm{ML^2T^{-2}/M = L^2T^{-2}}$. The fraction dE contained in the eddies with wavenumbers ranging from k to $k+dk$ is defined as

$$dE \;=\; E_k(k)\; dk.$$

It follows that the dimension of E_k is $\mathrm{L^3T^{-2}}$, and dimensional analysis prescribes:

$$E_k(k) \;=\; B\;\epsilon^{2/3}\;k^{-5/3}, \tag{8.8}$$

where B is a second dimensionless constant. It can be related to A of Equation (8.2) because the integration of $E_k(k)$ from $k_{\min} = \pi/L$ to $k_{\max} \sim \infty$ is the total energy in the system, which in good approximation is that contained in the largest eddies, namely $U^2/2$. Thus,

$$\int_{k_{\min}}^{\infty} E_k(k)\; dk \;=\; \frac{U^2}{2}\;, \tag{8.9}$$

from which follows

$$\frac{2}{3\pi^{2/3}}\;B \;=\; \frac{1}{2}\;A^2. \tag{8.10}$$

The value of B has been determined experimentally and found to be about 1.5 (Pope, 2000, page 231). From this, we estimate A to be 0.97.

The $-5/3$ power law of the energy spectrum has been observed to hold well in the *inertial range*, that is, for those intermediate eddy diameters that are remote from both largest and shortest scales. Figure 8.5 shows the superposition of a large number of longitudinal power spectra[2]. The straight line where most data overlap in the range $10^{-4} < k\nu^{3/4}/\epsilon^{1/4} < 10^{-1}$ corresponds to the $-5/3$ decay law predicted by the Kolmogorov turbulent cascade theory. The higher the Reynolds number of the flow, the broader the span of wavenumbers over which the $-5/3$ law holds.

[2]The longitudinal power spectrum is the spectrum of the kinetic energy associated with the velocity component in the direction of the wavenumber.

The few crosses at the top of the plot, which extend a set of crosses buried in the accumulation of data below, correspond to data in a tidal channel (Grant et al., 1962), for which the Reynolds number was the highest.

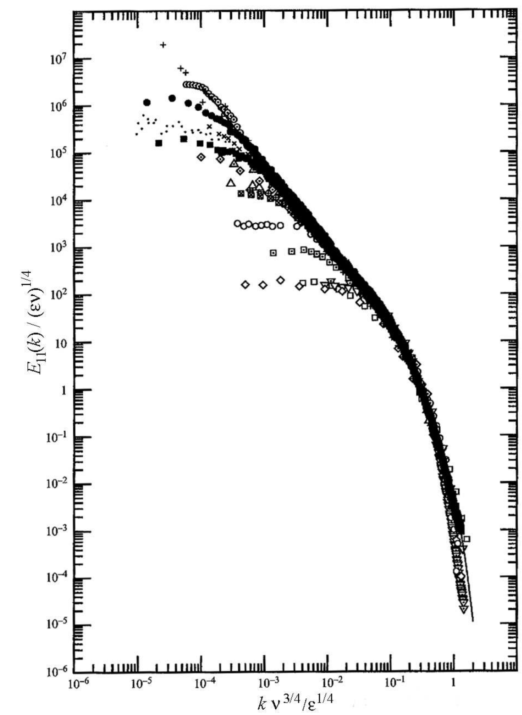

Figure 8.5: Longitudinal power spectrum of turbulence calculated from numerous observations taken outdoor and in the laboratory. [From Saddoughi and Veeravalli, 1994]

There is, however, some controversy over the $-5/3$ power law for E_k. Some investigators (Saffman, 1968; Long, 1997 and 2003) have proposed alternative theories that predict a -2 power law.

8.2 Shear-Flow Turbulence

Most environmental fluid systems are much shallower than they are wide. Such are the atmosphere, oceans, lakes and rivers. Their vertical confinement forces the flow to be primarily horizontal, and the vertical velocity, if any, is relatively weak. The ratio of vertical to horizontal velocity is typically on the order of the geometric

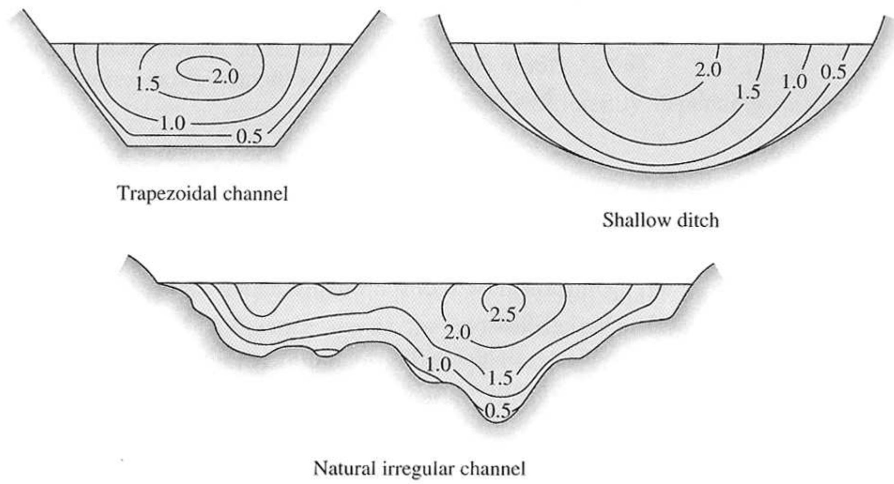

Figure 8.6: Velocity distribution across a sample of natural and artificial channels. [From White, 2003]

aspect ratio of the domain (depth over width), or less.

Unavoidable in such shallow–wide situation is friction between the main horizontal motion and the bottom boundary. Friction acts to reduce the velocity from some finite value in the interior of the flow to zero at the bottom, thus creating a vertical shear. Mathematically, if u is the velocity component in one of the horizontal directions and z the elevation above the bottom, then u is a function of z. The function $u(z)$ is called the *velocity profile* and its derivative du/dz, the *velocity shear*. Figure 8.6 sketches a few examples, which show velocity distributions across several water channels.

Environmental flows are invariably turbulent (high Reynolds number) and this greatly complicates the search for the velocity profile. As a consequence, much of what we know is derived from observations of actual flows, either in the laboratory or in nature.

The turbulent nature of the shear flow along a smooth or rough surface includes variability at short time and length scales, and the best observational techniques for the detailed measurements of these have been developed for the laboratory rather than outdoor situations. Figures 8.7 and 8.8 show the details of a turbulent flow along a smooth straight wall. Note the rolling over of fluid particles across the primary direction of the flow. Such cross-flow exchanges are responsible for momentum transfer in the direction perpendicular to the boundary, which sets the average velocity profile.

Numerous laboratory measurements of turbulent flows along smooth straight surfaces have led to the conclusion that the velocity varies solely with the stress τ_b exerted against the bottom, the fluid molecular viscosity μ, the fluid density ρ and,

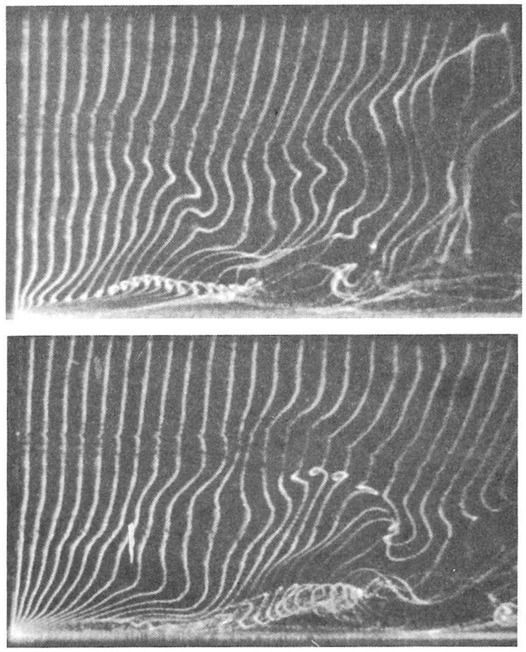

Figure 8.7: Near-wall structure of a turbulent shear flow visualized by hydrogen bubbles and viewed from the side. The pair of photographs shows the formation of a streamwise vortex in a matter of a few seconds. (From Kline et al., 1967)

of course, the distance z above the bottom. Thus,

$$u(z) \ = \ F(\tau_b, \mu, \rho, z).$$

Dimensional analysis permits the elimination of the mass dimension shared by τ_b, μ and ρ but not present in u and z, and we may write more simply:

$$u(z) \ = \ F\left(\frac{\tau_b}{\rho}, \frac{\mu}{\rho}, z\right).$$

The ratio μ/ρ is the kinematic viscosity ν (units of m^2/s), whereas the ratio τ_b/ρ has the same dimension as the square of a velocity (units of m^2/s^2). For convenience, it is customary to define

$$u_* \ = \ \sqrt{\frac{\tau_b}{\rho}}\ , \tag{8.11}$$

which is called the *friction velocity* or *turbulent velocity*. Physically, its value is related to the orbital velocity of the vortices that create the cross-flow exchange of particles and the momentum transfer.

The velocity structure thus obeys a relation of the form

$$u(z) \ = \ F(u_*, \nu, z),$$

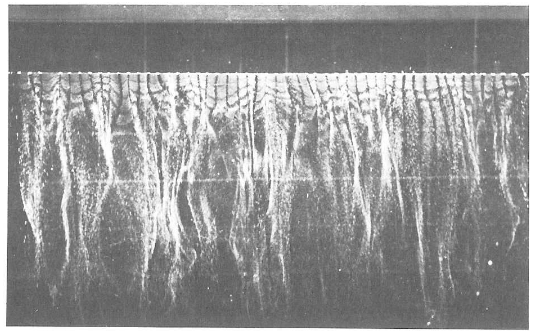

Figure 8.8: Top view of the same laboratory experiment as shown in Figure 8.7. The flow is directed from top to bottom in the photograph, and hydrogen bubbles are generated along the horizontal line. Hydrogen bubbles form streaks, which indicate a pattern of alternating convergence and divergence in the cross-stream direction. This pattern is the horizontal manifestation of streamwise vortives. (From Kline et al., 1967)

and further use of dimensional analysis allows us to reduce this to a function of a single variable:

$$\frac{u(z)}{u_*} \;=\; f\left(\frac{u_* z}{\nu}\right).$$

Logarithmic profile

The observational determination of the function f has been repeated countless times, every time with the same results, and it suffices here to provide a single report (Figure 8.9). When the velocity ratio u/u_* is plotted versus the logarithm of the dimensionless distance u_*z/ν, not only do all the points coalesce on a single curve, confirming that there is indeed no other variable to be invoked, but the curve also behaves as a straight line over a range of two orders of magnitude (from u_*z/ν between 10^1 and 10^3).

If the velocity is linearly dependent on the logarithm of the distance, then we can write for this portion of the velocity profile:

$$\frac{u(z)}{u_*} \;=\; A\;\ln\;\frac{u_* z}{\nu} \;+\; B.$$

Numerous experimental determinations of the constants A and B provide $A = 2.44$ and $B = 5.2$ within a 5% error (Pope, 2000). Tradition has it to write the function as:

$$u(z) \;=\; \frac{u_*}{\kappa}\;\ln\;\frac{u_* z}{\nu} \;+\; 5.2\;u_*, \tag{8.12}$$

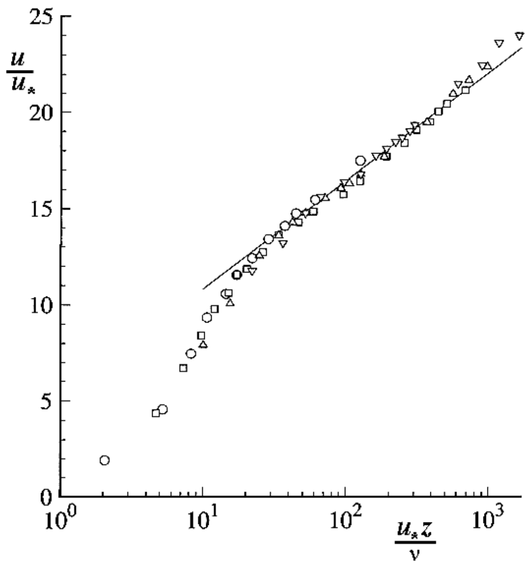

Figure 8.9: Mean velocity profiles in fully developed turbulent channel flow measured by Wei and Willmarth (1989) at various Reynolds numbers: circles $Re = 2970$, squares $Re = 14914$, upright triangles $Re = 22776$, and downright triangles $Re = 39582$. The straight line on this log-linear plot corresponds to the logarithmic profile of Equation (8.12). (From Pope, 2000)

where $\kappa = 1/A = 0.41$ is called the *von Kármán constant*[3]

The portion of the curve closer to the wall, where the logarithmic law fails, may be approximated by the laminar solution. Constant laminar stress $\nu du/dz = \tau_b/\rho = u_*^2$ implies $u(z) = u_*^2 z/\nu$ there. Ignoring the region of transition in which the velocity profile gradually changes from one solution to the other, we can attempt to connect the two. Doing so yields $u_* z/\nu = 11$. This sets the thickness of the laminar boundary layer δ as the value of z for which $u_* z/\nu = 11$, i.e.

$$\delta \;=\; 11\,\frac{\nu}{u_*}\;. \tag{8.13}$$

Most textbooks (e.g. Kundu, 1990) give $\delta = 5\nu/u_*$, for the region in which the velocity profile is strictly laminar, and label the region between $5\nu/u_*$ and $30\nu/u_*$ as the *buffer layer*, the transition zone between laminar and fully turbulent flow.

For water in ambient conditions, the kinetic molecular viscosity ν is equal to 1.0 $\times\ 10^{-6}$ m^2/s, while the friction velocity in a typical river rarely falls below 1 cm/s. This implies that δ can hardly exceed 1 mm in a river and is almost always smaller than the height of the cobbles, ripples and other asperities that typically line the bottom of the channel.

When this is the case, the velocity profile above the bottom asperities no longer depends on the molecular viscosity of the fluid but on the so-called *roughness height* z_o, such that

[3]in honor of Theodore von Kármán (1881–1963), Hungarian-born physicist and engineer who made significant contributions to fluid mechanics while working in Germany and who first introduced this notation.

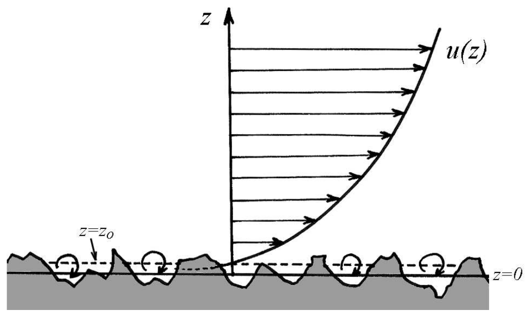

Figure 8.10: Velocity profile in the vicinity of a rough wall. The roughness heigh z_o is smaller than the averaged height of the surface asperities. So, the velocity u falls to zero somewhere within the asperities, where local flow degenerates into small vortices between the peaks, and the negative values predicted by the logarithmic profile are not physically realized.

$$u(z) \;=\; \frac{u_*}{\kappa}\,\ln\,\frac{z}{z_o}\;, \qquad (8.14)$$

as depicted in Figure 8.10. It is important to note that the roughness height is not the average height of bumps on the surface but is equal to a small fraction of it, about one tenth (Garratt, 1992, page 87). Table 8.1 lists a few values of environmental relevance. In this list, d_{90} is the particle diameter such that 90% of the particles have a smaller diameter than this.

Table 8.1 Typical values of the roughness height

	Type of surface	z_o (m)	Reference
Atmosphere			
	Sand, snow or sea	$0.016u_*^2/g$	Chamberlain (1983)
	Active sand sheet	0.00004	Draxler et al. (2001)
	Sand sheet	0.0004–0.0005	Draxler et al. (2001)
	Gravel lag	0.0002	Draxler et al. (2001)
	Bare soil	0.001–0.01	Garratt (1992)
	Flat open country	0.02–0.06	Davenport (1965)
	Grass - sparse	0.0012	Clarke et al. (1971)
	Grass - thick	0.026	Su et al. (2001)
	Grass - thin	0.05	Sutton (1953)
	Wheat field	0.015	Garratt (1977)
	Corn field	0.064	Kung (1961)
	Vines - along rows	0.023	Hicks (1973)
	Vines - across rows	0.12	Hicks (1973)
	Vegetation	0.2	Fichtl and McVehil (1970)
	Savannah	0.4	Garratt (1980)
	Scrub	0.049	Su et al. (2001)
	Trees	0.4	Fichtl and McVehil (1970)
	Trees - sparse	0.1 of tree height	Guyot and Seguin (1978)
	Forest - temperate	0.28–0.92	Hicks et al. (1975), Thom et al. (1975), Jarvis et al. (1976)
	Forest - tropical	2.2–4.8	Thomson and Pinker (1975), Shuttleworth (1989)
	Residential neighborhood	1.0–5.0	Davenport (1965)
	City with high rises		
	Hilly terrain	1–10	
Water			
	Concrete channel	0.0003–0.003	Chanson (2004)
	Mud	0.0	
	Coarse grains	$2.0\ d_{90}$	Kamphuis (1974)
	Rocks	0.05	

Drag coefficient

The average velocity in a vertically confined domain can be related to the bottom stress. For a rough bottom at $z = 0$ and a free surface at $z = h$, the average velocity is given by:

$$\begin{aligned} \bar{u} &= \frac{1}{h} \int_0^h u(z)dz = \frac{u_*}{\kappa h} \int_0^h \ln \frac{z}{z_o}\, dz \\ &= \frac{u_*}{\kappa} \left[\ln\left(\frac{h}{z_o}\right) - 1 \right], \end{aligned} \tag{8.15}$$

which permits to relate the friction velocity u_* to the average velocity $\bar{u}$:

$$u_* = \frac{\kappa \bar{u}}{\ln(h/z_o) - 1} . \tag{8.16}$$

This in turn provides the relationship between the bottom stress and the average velocity:

$$\tau_b \ = \ \rho u_*^2 \ = \ \frac{\rho \kappa^2 \bar{u}^2}{[\ln(h/z_{\rm o}) - 1]^2} \ . \tag{8.17}$$

If we introduce a drag coefficient C_D such that $\tau_b = C_D \rho \bar{u}^2$, then its value is:

$$C_D \ = \ \kappa^2 \left[\ln\left(\frac{h}{z_{\rm o}}\right) - 1 \right]^{-2} . \tag{8.18}$$

Note that the drag coefficient is not a constant but depends on the ratio of the fluid depth to the roughness height. For $\kappa = 0.41$ and a ratio $h/z_{\rm o}$ in the range 50–1000, the drag coefficient varies between 0.005 and 0.020.

8.3 Mixing Length

To solve more general problems in turbulence, an attempt has been made to assimilate the mixing caused by turbulence to an enhanced viscosity. This amounts to a search for a *turbulent viscosity* (often called an *eddy viscosity*) that would replace in turbulent flows the molecular viscosity of laminar flows. In analogy with Newton's law for viscous fluids, which has the tangential stress τ proportional to the velocity shear du/dz with the coefficient of proportionality being the molecular viscosity μ, one writes for turbulent flow:

$$\tau \ = \ \mu_T \, \frac{du}{dz} \ , \tag{8.19}$$

where the turbulent viscosity μ_T supersedes the molecular viscosity μ.

For the logarithmic profile (8.14) of a flow along a rough surface, the velocity shear is $du/dz = u_*/\kappa z$ and the stress τ is uniform across the flow (for lack of acceleration and of other forces): $\tau = \tau_b = \rho u_*^2$, giving

$$\rho u_*^2 \ = \ \mu_T \, \frac{u_*}{\kappa z}$$

and thus

$$\mu_T \ = \ \rho \kappa z u_* . \tag{8.20}$$

Note that unlike the molecular viscosity, the turbulent viscosity is not constant in space, for it is not a property of the fluid but of the flow, including its geometry. The corresponding turbulent *kinematic* viscosity is

$$\nu_T \ = \ \frac{\mu_T}{\rho} \ = \ \kappa z u_* ,$$

which can be expressed as the product of a length by the turbulent velocity:

$$\nu_T \;=\; l_m u_*, \tag{8.21}$$

with the *mixing length* l_m defined as

$$l_m \;=\; \kappa z, \tag{8.22}$$

for the turbulent flow along a rough boundary.

To generalize to turbulent flows other than the logarithmic profile, we keep Equation (8.21) but intend to adapt the mixing length l_m and turbulent velocity u_* to every situation. The turbulent velocity is replaced by means of the velocity shear by reasoning that it is the shear that creates flow instabilities, induces turbulence and thus creates a turbulent viscosity. In other words, the greater the velocity shear is, the larger the turbulent viscosity ought to be. Let us then eliminate the turbulent velocity u_* in favor of the velocity shear du/dz and then generalize the latter for flows other than parallel flow, as follows:

$$\tau \;=\; \mu_T \,\frac{du}{dz} \;=\; \rho \nu_T \,\frac{du}{dz}$$

with the local turbulent velocity defined from the magnitude of the local stress by $|\tau| = \rho u_*^2$ and $\nu_T = l_m u_*$ according to (8.21):

$$\rho u_*^2 \;=\; \rho l_m u_* \left|\frac{du}{dz}\right| \qquad \longrightarrow \qquad u_* \;=\; l_m \left|\frac{du}{dz}\right|$$

yielding

$$\nu_T \;=\; l_m u_* \;=\; l_m^2 \left|\frac{du}{dz}\right|. \tag{8.23}$$

In arbitrary, three-dimensional turbulent flows there are several components to the velocity shear, actually an entire array. Smagorinsky (1963) proposed the following extension. First, the *rate-of-strain tensor* is defined, with components

$$S_{ij} \;=\; \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} \;+\; \frac{\partial u_j}{\partial x_i}\right) \tag{8.24}$$

where i and j are indices running from 1 to 3, such that $(x_1,\, x_2,\, x_3) = (x,\, y,\, z)$ and $(u_1,\, u_2,\, u_3) = (u,\, v,\, w)$. From the tensor S_{ij} is defined the overall strain S by

$$S^2 \;=\; 2 \sum_{i=1}^{3} \sum_{j=1}^{3} S_{ij}^2 \tag{8.25}$$

that supersedes the earlier velocity shear du/dz. According to this model of turbulence, the turbulent kinematic viscosity is

$$\nu_T \;=\; l_m^2 S. \tag{8.26}$$

But, this way of proceeding includes not only the actual velocity shear components (such as $\partial u/\partial y$, $\partial v/\partial x$, etc.) but also convergence/divergence terms ($\partial u/\partial x$,

$\partial v/\partial y$ and $\partial w/\partial z$) that do not contribute to instabilities and turbulence. A remedy to this situation is the model of Baldwin and Lomax (1978) in which the *rate-of-rotation tensor*

$$\Omega_{ij} \;=\; \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} \;-\; \frac{\partial u_j}{\partial x_i}\right) \tag{8.27}$$

is used instead of the rate-of-strain tensor. All non-zero elements of this tensor correspond to vorticity components. From the tensor Ω_{ij} is defined the overall shear Ω by

$$\Omega^2 \;=\; 2\sum_{i=1}^{3}\sum_{j=1}^{3}\Omega_{ij}^2 \tag{8.28}$$

and, according to this model, the turbulent kinematic viscosity is

$$\nu_T \;=\; l_m^2\,\Omega. \tag{8.29}$$

It remains now to say something about the mixing length l_m. Since $l_m = \kappa z$ for parallel flow along a rough boundary, where z is the distance to the boundary, one possibility is to take for the mixing length the product of the von Kármán constant $\kappa = 0.41$ by the distance to the nearest wall. For a flow between two parallel boundaries, like a river between its two banks (at $y = 0$ and $y = W$), a choice is

$$l_m \;=\; \kappa\,\frac{y(W-y)}{W}\;. \tag{8.30}$$

8.4 Turbulence in Stratified Fluids

The presence of stratification induces buoyancy forces against which turbulent motions need to work. The result is a subdued form of turbulence. Turner (1973, Chapter 5), Thorpe (review, JGR, 1987), Tritton (1988, Section 21.7).

Ozmidov buoyancy vertical scale

$$L_b \;=\; \sqrt{\frac{\epsilon}{N^3}} \tag{8.31}$$

in which ϵ is the energy dissipation rate per unit mass and N is the stratification frequency.

8.5 Two-dimensional Turbulence

Because of their small vertical-to-horizontal aspect ratio, environmental systems exhibit in the horizontal a form of turbulence that is nearly two-dimensional.

8.6 Closure Schemes

Since no complete theory of turbulence exists, there is a need to distill somehow the results of observations into some empirical rules. A computer simulation model that incorporates one or several of these rules is said to include a closure scheme. A large number of closure schemes have been proposed over the years, with varying degrees of success. We present here only a couple of them, which have each been tested extensively in the context of environmental systems.

$k - \epsilon$ model.

Mellor and Herring (1973), Mellor and Yamada (1982).

8.7 Large-Eddy Simulations

Pope (2000, Chapter 13)

Problems

8-1. What would be the energy spectrum $E_k(k)$ in a turbulent flow where all length scales were contributing equally to dissipation? Is this spectrum realistic?

8-2. The earth receives 1.75×10^{17} W from the sun, and we can assume that half of this energy input is being dissipated in the atmosphere, with the rest going to land and sea.
(a) Using the known ground-level pressure and surface area of the earth, determine the mass of the atmosphere.
(b) Estimate the rate of energy dissipation ϵ in the atmosphere.
(c) What is the Reynolds number of the atmosphere?
(c) Finally, estimate the smallest eddy scale in the air and its ratio to the largest scale.
Useful numbers: Standard atmospheric pressure is 101,325 Pa, the radius of the earth is 6,371 km, and the molecular viscosity of air at ambient temperature and pressure is $\nu = 1.51 \times 10^{-5}$ m^2/s.

8-3. There are applications in turbulent boundary-layer flow for which the logarithmic function creates mathematical complications and it is desirable to use a power law to represent the velocity profile, such as

$$
\begin{aligned}
u(z) &= U \left(\frac{z}{d}\right)^{\alpha} \quad \text{for } z < d \\
&= U \quad \text{for } d \leq z,
\end{aligned}
$$

with d being a suitable boundary-layer thickness. Determine the values of d and the exponent α for best fit over the interval $10z_o < z < 2000z_o$. What is then the relation between the far-field velocity U and the friction velocity u_*? And, from this, establish an approximate drag law relating wall stress to far field velocity.

8-4. A 20-m deep reservoir has a river throughflow sustaining a steady current varying from 12 cm/s near the surface to zero at the bottom. At the time of these observations, a wind was blowing that imposed a stress of 0.1 N/m^2 on the water surface. Compare the turbulence activity generated by the wind stress to that maintained by the river flow. Which one by itself leads to the largest dissipation rate? What is the Kolmogorov dissipation scale in the reservoir?

8-5.

8-6.

8-7.