Empirische Untersuchung über den Verlauf der zahlentheoretischen Function $\sigma(n) = \sum_{x=1}^{x=n} \mu(x)$ im Intervalle von 0 bis 150000

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Empirical Investigation of the behavior of the number-theoretic function $\sigma(n) = \sum_{x=1}^{x=n} \mu(x)$ in the interval from 0 to 150000.

We denote by $\mu(x)$ the well known number-theoretical function whose value is 1 or -1 when its argument x (a whole number) has respectively an even or odd number of distinct prime factors, and whose value is 0 for all arguments which are divisible by a square (bigger than 1).

The function which is derived from this one through summation,

$$\sigma(n) = \sum_{x=1}^{x=n} \mu(x),$$

is of particular importance in connection with asymptotic laws in number theory; however, it is has so far not been possible to determine a lower bound for the absolute value of this function that is sufficient to establish these asymptotic laws. Thus it appeared to me to be desirable to get to know empirically the behavior of the function $\sigma(n)$ over a fairly large finite interval, and I am permitting myself to present the results of this investigation in the following tables.

...

A perusal of the tables (or alternatively, a glance at the graph) reveals that as a result of this investigation, we can immediately recognize the fact, which was already observed for smaller intervals, that the absolute value of the function $\sigma(n)$ never achieves the value $\sqrt{n}$. Only for $n = 1$ does this law suffer an exception, in that $\sigma(1) = 1$. This law, so important for number theory, is true in a strong sense up to $n = 150000$. The function $\sigma(n)$ oscillates in ever larger waves about the horizontal axis. A rule for the growth of these waves is hard to formulate; However, it is noticable that the quotient $\frac{\sigma(n)}{\sqrt{n}}$ remains reasonably constant in absolute value at its occasional extreme values.

...

As we see, the absolute value of the function $\sigma(n)$ does not once achieve the bound $1/2\sqrt{n}$ in the interval under investigation. [1] Consequently, if any conjecture about the behavior of this function outside this interval is indicated, it is that it appears that the relation

$$|\sigma(n)| < \sqrt{n}$$

is generally valid.
[2]

[1] Just at the beginning, in the neighborhood of $n = 200$ there are places where $|\sigma(n)| > 1/2\sqrt{n}$. But even in these places, the relation $|\sigma(n)| < \sqrt{n}$ remains true.

[2] If we compare the series of non-0 values of $\mu(n)$ with a purely random sequence of positive and negative units (such as one might produce, for example, with a die having $+1$ on three faces and -1 on the other three faces), it appears that the absolute value of $\sigma(n)$ in the interval under investigation stays beneath what we would expect for the difference between the number of positive and negative units in such a random process. The most likely value of this difference is easy to determine, if we use for a measure the arithmetic mean of the absolute value of the deviation over all possible sequences.
... (derivation omitted) ...

$$\frac{M - \sqrt{m}}{\sqrt{\frac{\pi}{2}}.}$$

If we consider the place n in our tables, we will find up to that point roughly $\frac{6}{\pi^2 n}$ places where $\mu(n)$ is different from 0; we therefore have

$$m = \frac{6}{\pi^2}n;$$

we obtain from this through supstitution the values

$$M = \sqrt{\frac{12}{\pi^2}}\sqrt{n} = 0.62211\sqrt{n}.$$

As we see, the magnitude of the expected deviation of a purely random sequence is still on the order of $\sqrt{n}$. The coefficient is however greater than even the maximal value of $\frac{|\sigma(n)|}{\sqrt{n}}$, and consequently greater by far than the average value of this quotient.