Electromagnetic Induction (EMI) Response from Conducting and Permeable Spheroidal Shells

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Presentation Outline

Introduction – Motivation

Exact Solution
- Formulation
- Boundary conditions
- Truncated solution

Results and limiting cases for small and moderate size parameter $c$

Conclusion
EMI is promising in the detection of Unexploded Ordnance (UXO)

- Very low frequencies (3Hz-30KHz)
- Low frequency permits ignoring ground effects
- Spheroidal geometry allows models of complex shapes
  - Spheres
  - Plates
  - Needles
- Can discriminate between different shapes and compositions
Geometry and Formulation

Spheroidal shell EMI response

- conducting and permeable shell
- time-harmonic excitation
- quasi-magnetostatic regime

Region I

\[
\begin{align*}
\mu_1, \sigma_1, \overline{H}_1 \\
k_1^2 &= i \omega_1 \sigma_1 \mu_1 \\
\nabla \times \nabla \times \overline{H}_1 - k_1^2 \overline{H}_1 &= 0
\end{align*}
\]

Region II

\[
\begin{align*}
\mu_2 &\approx \mu_0, \sigma_2 \approx 0 \\
k_2^2 &\approx 0 \\
\overline{H}_2 &= -\nabla U_2
\end{align*}
\]

Region III

\[
\begin{align*}
\mu_3, \sigma_3, \overline{H}_3 \\
k_3^2 &= i \omega_3 \sigma_3 \mu_3 \\
\nabla \times \nabla \times \overline{H}_3 - k_3^2 \overline{H}_3 &= 0
\end{align*}
\]

\[
\overline{H}_2 = \overline{H}_o + \overline{H}_s = -\nabla U_o - \nabla U_s
\]
Exact Solution – Regions I and III

Expand $\bar{H}_1$ and $\bar{H}_3$ in terms of vector spheroidal wave functions.

$$\bar{H}_1 = H_o \sum_{m=0}^{\infty} \sum_{n=m}^{\infty} \sum_{p=0}^{1} \left[ A^{(M)}_{pmn} \overline{M}^{r(1)}_{pmn}(c; \xi, \eta, \phi) + A^{(N)}_{pmn} \overline{N}^{r(1)}_{pmn}(c; \xi, \eta, \phi) ight.$$

$$+ C^{(M)}_{pmn} \overline{M}^{r(3)}_{pmn}(c; \xi, \eta, \phi) + C^{(N)}_{pmn} \overline{N}^{r(3)}_{pmn}(c; \xi, \eta, \phi) \right]$$

$$\bar{H}_3 = H_o \sum_{m=0}^{\infty} \sum_{n=m}^{\infty} \sum_{p=0}^{1} \left[ D^{(M)}_{pmn} \overline{M}^{r(1)}_{pmn}(c; \xi, \eta, \phi) + D^{(N)}_{pmn} \overline{N}^{r(1)}_{pmn}(c; \xi, \eta, \phi) \right]$$

where

- $H_o \Rightarrow$ primary field strength
- $\xi, \eta, \phi \Rightarrow$ spheroidal coordinates
- $c = kd/2 \Rightarrow$ spheroidal size parameter
- $d = 2\sqrt{b^2 - a^2} \Rightarrow$ interfocal distance
- $A^{(M)}_{pmn}, A^{(N)}_{pmn}, C^{(M)}_{pmn}, C^{(N)}_{pmn} \Rightarrow$ expansion coefficients
Exact Solution – Region II

Expand potentials $U_o$, $U_s$, and $U_2$ as

\[
U_o(\mathbf{r}) = \frac{H_0 d}{2} \sum_{m=0}^{\infty} \sum_{n=m}^{\infty} \sum_{p=0}^{1} b_{pmn} \Phi_{pmn}^{(1)}(\xi, \eta, \phi)
\]

\[
U_s(\mathbf{r}) = \frac{H_0 d}{2} \sum_{m=0}^{\infty} \sum_{n=m}^{\infty} \sum_{p=0}^{1} B_{pmn} \Phi_{pmn}(\xi, \eta, \phi)
\]

\[
U_2 = U_o + U_s
\]

where

\[
\Phi_{pmn}^{(1)}(\xi, \eta, \phi) = P_n^m(\xi)P_n^m(\eta)T_{pm}(\phi)
\]

\[
\Phi_{pmn}(\xi, \eta, \phi) = Q_n^m(\xi)P_n^m(\eta)T_{pm}(\phi)
\]
Solutions in Terms of Spheroidal Wavefunctions

**Unknowns** – $A_{pmn}^{(M)}, A_{pmn}^{(N)}, C_{pmn}^{(M)}, C_{pmn}^{(N)}, D_{pmn}^{(M)}, D_{pmn}^{(N)},$ and $B_{pmn}$

**Scalar spheroidal wave function of the $i^{th}$ kind is**

$$
\psi_{pmn}^{(i)} = S_{mn}(c, \eta) R_{mn}^{(i)}(c, \xi) T_{pm}(\phi)
$$

where

- $S_{mn}(c, \eta) \Rightarrow$ spheroidal angle function ($\rightarrow$ Sum of Associated Legendre functions)
- $R_{mn}^{(i)}(c, \xi) \Rightarrow$ spheroidal radial function ($\rightarrow$ Sum of Spherical Bessel functions)
- $T_{pm}(\phi) \Rightarrow$ spheroidal azimuthal function ($\rightarrow$ sines and cosines)

**Vector spheroidal wave functions are generated from the scalar wave function**

$$
\overline{M}_{pmn}^{r(i)} = \nabla \psi_{pmn}^{(i)} \times \hat{r}
$$

$$
\overline{N}_{pmn}^{r(i)} = \frac{1}{k_1} \nabla \times \overline{M}_{pmn}^{r(i)}
$$
Boundary Conditions

Outer ($\alpha$) boundary $d = d_{\alpha}$

\[ H_1 \eta = H_2 \eta \]
\[ H_1 \phi = H_2 \phi \]
\[ \mu_r H_1 \xi = H_2 \xi \]

Matching $H_\xi$

\[ \mu_r \sum_{n=m}^{\infty} \left\{ (-1)^p m A_{p,n}^{(M)} \bar{M}_{\xi;n}^{(1)}(c_{1\alpha}, \eta) + \frac{1}{c_{1\alpha}} A_{p,n}^{(N)} \bar{N}_{\xi;n}^{(1)}(c_{1\alpha}, \eta) \right\} + \mu_r \sum_{n=m}^{\infty} \left\{ (-1)^p m C_{p,n}^{(M)} \bar{M}_{\xi;n}^{(3)}(c_{1\alpha}, \eta) + \frac{1}{c_{1\alpha}} C_{p,n}^{(N)} \bar{N}_{\xi;n}^{(3)}(c_{1\alpha}, \eta) \right\} = - \sum_{n=m}^{\infty} \left[ b_{p,n} \frac{dP_n}{d\xi_0} + B_{p,n} \frac{dQ_n}{d\xi_0} \right] P_n(\eta) \]

After “Testing”

\[ \mu_r \sum_{n=m}^{\infty} \left\{ (-1)^p m A_{p,n}^{(M)} I_{\xi;n}^{(1)}(c_{1\alpha}, \eta) + \frac{1}{c_{1\alpha}} A_{p,n}^{(N)} I_{\xi;n}^{(1)}(c_{1\alpha}, \eta) \right\} + \mu_r \sum_{n=m}^{\infty} \left\{ (-1)^p m C_{p,n}^{(M)} I_{\xi;n}^{(3)}(c_{1\alpha}, \eta) + \frac{1}{c_{1\alpha}} C_{p,n}^{(N)} I_{\xi;n}^{(3)}(c_{1\alpha}, \eta) \right\} = - \left[ b_{p,n} \frac{dP_n}{d\xi_0} + B_{p,n} \frac{dQ_n}{d\xi_0} \right] \]

Inner ($\beta$) boundary $d = d_{\beta}$

\[ H_1 \eta = H_3 \eta \]
\[ H_1 \phi = H_3 \phi \]
\[ \mu_r H_1 H_3 = \mu_r H_3 \xi \]

Matching $H_\xi$

\[ \mu_r \sum_{n=m}^{\infty} \left\{ (-1)^p m A_{p,n}^{(M)} \bar{M}_{\xi;n}^{(1)}(c_{1\beta}, \eta) + \frac{1}{c_{1\beta}} A_{p,n}^{(N)} \bar{N}_{\xi;n}^{(1)}(c_{1\beta}, \eta) \right\} + \mu_r \sum_{n=m}^{\infty} \left\{ (-1)^p m C_{p,n}^{(M)} \bar{M}_{\xi;n}^{(3)}(c_{1\beta}, \eta) + \frac{1}{c_{1\beta}} C_{p,n}^{(N)} \bar{N}_{\xi;n}^{(3)}(c_{1\beta}, \eta) \right\} \]

After “Testing”

\[ \mu_r \sum_{n=m}^{\infty} \left\{ (-1)^p m A_{p,n}^{(M)} J_{\xi;n}^{(1)}(c_{3\beta}, \eta) + \frac{1}{c_{3\beta}} A_{p,n}^{(N)} J_{\xi;n}^{(1)}(c_{3\beta}, \eta) \right\} + \mu_r \sum_{n=m}^{\infty} \left\{ (-1)^p m C_{p,n}^{(M)} J_{\xi;n}^{(3)}(c_{3\beta}, \eta) + \frac{1}{c_{3\beta}} C_{p,n}^{(N)} J_{\xi;n}^{(3)}(c_{3\beta}, \eta) \right\} \]
Truncated Solution

Truncate infinite series at $L_T$

Recast into matrix equations

\begin{align*}
\overline{Z}^{(1)}_{\xi}(c_{1\alpha}, \eta) \cdot \overline{A} + \overline{Z}^{(3)}_{\xi}(c_{1\alpha}, \eta) \cdot \overline{C} & = \overline{W}^{(p)}_{\xi} \cdot \overline{b} + \overline{W}^{(q)}_{\xi} \cdot \overline{B} \\
\overline{Z}^{(1)}_{\phi}(c_{1\alpha}, \eta) \cdot \overline{A} + \overline{Z}^{(3)}_{\phi}(c_{1\alpha}, \eta) \cdot \overline{C} & = \overline{W}^{(p)}_{\phi} \cdot \overline{b} + \overline{W}^{(q)}_{\phi} \cdot \overline{B} \\
\overline{Z}^{(1)}_{\eta}(c_{1\alpha}, \eta) \cdot \overline{A} + \overline{Z}^{(3)}_{\eta}(c_{1\alpha}, \eta) \cdot \overline{C} & = \overline{W}^{(p)}_{\eta} \cdot \overline{b} + \overline{W}^{(q)}_{\eta} \cdot \overline{B} \\
\overline{Z}^{(1)}_{\xi}(c_{1\beta}, \eta) \cdot \overline{A} + \overline{Z}^{(3)}_{\xi}(c_{1\beta}, \eta) \cdot \overline{C} & = \overline{Z}^{(1)}_{\xi}(c_{3\beta}, \eta) \cdot \overline{D} \\
\overline{Z}^{(1)}_{\phi}(c_{1\beta}, \eta) \cdot \overline{A} + \overline{Z}^{(3)}_{\phi}(c_{1\beta}, \eta) \cdot \overline{C} & = \overline{Z}^{(1)}_{\phi}(c_{3\beta}, \eta) \cdot \overline{D} \\
\overline{Z}^{(1)}_{\eta}(c_{1\beta}, \eta) \cdot \overline{A} + \overline{Z}^{(3)}_{\eta}(c_{1\beta}, \eta) \cdot \overline{C} & = \overline{Z}^{(1)}_{\eta}(c_{3\beta}, \eta) \cdot \overline{D}
\end{align*}

where

- $\overline{Z}^{(i)}_{\beta}(\xi)$ are $L_T \times 2L_T$ matrices and
- $\overline{W}^{(p,q)}_{\beta}(\xi) (\beta = \xi, \eta, \phi)$ are $L_T \times L_T$ matrices, respectively
Far-Field Dipole Response

Far-field expression for secondary field

- Axial

\[ M_z = H_o \left( \frac{\pi d^3}{6} \right) B_{001} \]

- Transverse

\[ M_t = H_o \left( \frac{\pi d^3}{3} \right) B_{011} \]
Results
Prolate Spheroid, Axial primary field

- elongation \((e) = 1.001\)
- size ratio \(\frac{d_\beta}{d_\alpha} = 0.01\)

\[\mu_{r1} = \mu_{r2} = 1\]

wavenumber ratio \(k_3/k_1 = 1\)

Induction number \(\log(k_1 a)\)

Normalized EMI response

Re\{analytic shell\}, Im\{analytic shell\}, Re\{analytic solid\}, Im\{analytic solid\}, Re\{SPA solid\}, Im\{SPA solid\}
Prolate Spheroid, Axial primary field

- elongation $(e) = 1.001$
- size ratio $\frac{d_\beta}{d_\alpha} = 0.01$
- $\mu_{r1} = \mu_{r2} = 1$
- wavenumber ratio $\frac{k_3}{k_1} = 0.1$

Normalized EMI response versus induction number $\log(k_1 a)$.
Prolate Spheroid, Axial primary field

- Elongation \( (e) = 1.001 \)
- Size ratio \( \frac{d_\beta}{d_\alpha} = 0.01 \)
- \( \mu_{r1} = \mu_{r2} = 10 \)
- Wavenumber ratio \( \frac{k_3}{k_1} = 0.1 \)

Diagram showing the normalized EMI response against the induction number \( \log(k_1 a) \). The graph compares different analytic and solid models, including real and imaginary parts for each case.
Prolate Spheroid, Axial primary field

- elongation (e) = 3
- size ratio $d_\beta/d_\alpha = 0.01$
- $\mu_{r1} = \mu_{r2} = 1$
- wavenumber ratio $k_3/k_1 = 0.1$

Induction number $\log(k_1 a)$

Normalized EMI response

- $\text{Re}\{\text{analytic shell}\}$
- $\text{Im}\{\text{analytic shell}\}$
- $\text{Re}\{\text{analytic solid}\}$
- $\text{Im}\{\text{analytic solid}\}$
- $\text{Re}\{\text{SPA solid}\}$
- $\text{Im}\{\text{SPA solid}\}$
Prolate Spheroid, Axial primary field

- Elongation: (e) = 1.001
- Size ratio: $\frac{d_\beta}{d_\alpha} = 0.4$
- Permeability: $\mu_{\alpha} = \mu_{\beta} = 1$
- Wavenumber ratio: $\frac{k_3}{k_1} = 1$

Normalized EMI response vs. Induction number $\log(k_1a)$
Prolate Spheroid, Axial primary field

 elongation (e) = 1.001
 size ratio \( \frac{d_\beta}{d_\alpha} = 0.4 \)

 \( \mu = \mu = 1 \)

 wavenumber ratio \( \frac{k_3}{k_1} = 0.25 \)

\[
\begin{align*}
\text{Normalized EMI response} & \\
\text{Induction number } \log(k_1 a) & \\
\end{align*}
\]
Prolate Spheroid, Axial primary field

- Elongation \( (e) = 1.001 \)
- Size ratio \( \frac{d_\beta}{d_\alpha} = 0.8 \)
- \( \mu_{r1} = \mu_{r2} = 1 \)
- Wavenumber ratio \( \frac{k_3}{k_1} = 0.25 \)

Induction number \( \log(k_1 a) \) vs. Normalized EMI response.

Graph shows data for \( \text{Re}\{\text{analytic shell}\} \), \( \text{Im}\{\text{analytic shell}\} \), \( \text{Re}\{\text{analytic solid}\} \), \( \text{Im}\{\text{analytic solid}\} \), \( \text{Re}\{\text{SPA solid}\} \), and \( \text{Im}\{\text{SPA solid}\} \).
Prolate Spheroid, Axial primary field

 elongation \( (e) = 1.001 \)
 size ratio \( \frac{d_β}{d_α} = 0.4 \)
\[ \mu_r = \mu_r = 10 \]
 wavenumber ratio \( \frac{k_3}{k_1} = 0.25 \)
Prolate Spheroid, Axial primary field

- Elongation \((e) = 3\)
- Size ratio \(\frac{d_\beta}{d_\alpha} = 0.4\)
- \(\mu_r = \mu = 10\)
- Wavenumber ratio \(\frac{k_3}{k_1} = 0.5\)

Graph showing normalized EMI response vs. induction number \(\log(k_1a)\).
Conclusion

Full analytic solution for the Electromagnetic Induction (EMI) response of a conducting and permeable spheroidal shell demonstrated

Spheroidal shells provide flexible model for various canonical shapes modeling UXOs including solid and hollow spheres, needles, and disks.

Method becomes unstable/ill-conditioned for high large size parameter $c$

High frequency techniques such as the Small Perturbation Approximation (SPA) and Asymptotic approximations for $S_{mn}(c, \eta)$ and $R_{mn}^{(i)}(c, \xi)$ may help overcome this limitation.

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