Today we’re going to use the think-pair-share technique. We have several goals for today:

1. Students will be able to articulate why there are $2^n$ elements in the power set of a set with $n$ elements.

2. Students will be able to state what $\binom{n}{k}$ means and will be able to compute it.

3. Students will be able to state what a factorial is and will be able to compute it.

First, I need your help today. Please make a tent from a piece of paper like so with your name on it and place your tent on your desk. Second, when I ask questions, please do not raise your hands. Instead, I will ask one of you to answer.

Let’s begin by summarizing your experiments from last time. Note: We’re going to fill in the following table. I’ve listed it first as it will be without their answers, then after the experiment discussion, and lastly, after we extend what they learned in the experiments.

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How many ways did you find to fill \{2, 3, 4, 5\} boxes with stars and bars? (Asked as four separate questions)

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1. What if we had one box? Take a few seconds to think about this and consult one neighbor.

2. What if we have 6 boxes? Again take a few seconds to think about this and consult one neighbor. Also talk with your neighbor about why you think you’re correct.
   (a) How did you figure it out?
   (b) Did anyone have a different idea?

3. (If the idea of a pattern is not brought up, ask the following:) Does anyone see a pattern? Take some time to think about this and consult one neighbor.

4. So for any number of boxes, how many different ways can we fill the boxes with stars and bars? Take a few moments to think about this and consult one neighbor.

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<td></td>
<td>2</td>
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<td>$2^n$</td>
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Recalling that the box activity and the light switch activity were similar, for each box or light switch, we have two choices: star or candy bar, or on or off. Therefore if there are 10 boxes or light switches, we have $2^{10}$ ways to choose for the boxes and the light switches.
There is a third way that we could view our light switches and boxes. We could have a set of objects labeled $A, B$. We can then make a list of all possible sets of those objects.

$$\emptyset, A, B, \{A, B\}$$

Suppose we have a set of objects labeled $A, B, C$. Take a moment and write out that list with a partner:

$$\emptyset, A, B, C, \{A, B\}, \{A, C\}, \{B, C\}, \{A, B, C\}$$

This list is called the *Power set* of the set $S = A, B, C$. It is the set of all subsets of S.

Thinking about the light switches, we can label three of them $A, B, C$ and turn the light labeled $A$ on if $A$ is included.

Let’s suppose that we have 5 boxes.

1. If we don’t use any candy bars, how many stars do we need to use to fill the boxes? And how many ways can we fill the boxes using only stars? Take some time to think about this and consult *one* neighbor.

   *Ans:* 1

2. If we use only 1 candy bar, how many stars do we need to use to fill the boxes? And how many ways can we fill the boxes using 1 candy bar and four stars? Take some time to think about this and consult *one* neighbor.

   *Ans:* 5 *Ask as a follow-up:* How did you figure this out?

   In this situation, we know that we’re only using one candy bar. So we first chose where we will place the candy bar and then can fill in the stars in the remaining boxes.

   When working in this framework, we can use the following notation $\binom{n}{k}$ and say “$n$ choose $k$”. For the two situations we have just discussed, we have $\binom{5}{0}$, and $\binom{5}{1}$. Or for five boxes choose 0, or five boxes choose 1.

3. What if we have four boxes? What are $\binom{4}{0}$ and $\binom{4}{1}$? Take a moment and talk this over with one or two neighbors.

4. What about for three boxes? What are $\binom{3}{0}$, and $\binom{3}{1}$? Take a moment and talk this over with one or two neighbors.

5. What patterns do we see emerging? For any number $n$, what is $\binom{n}{0}$?

6. For any number $n$, what is $\binom{n}{1}$?

7. Returning to our boxes, stars, and candy bars, if we use 2 candy bars, how many stars do we need to use to fill the boxes? And how many ways can we fill the boxes using 2 candy bars and three stars? Take some time to think about this and consult *one* neighbor.

   *Ans:* 10 *Ask as a follow-up:* How did you figure this out?
In this situation, we know that we’re using two candy bars. So we first chose where we will place the candy bars and then can fill in the stars in the remaining boxes. How do we write this?

8. For any number $n$, what is $\binom{n}{2}$? Try this for $n = 3$, $n = 4$ and $n = 5$. This one is a bit trickier and is often referred to as the “handshake problem.”

Let’s begin by introducing the factorial. When we say “$n$ factorial, written $n!$, we mean $n \times (n-1) \times (n-2) \times \ldots \times 3 \times 2 \times 1$.

9. What is $3!$? What is $4!$?

10. We can rewrite $\binom{n}{k} = \frac{n!}{k!(n-k)!}$. Example: $\binom{5}{2}$. Using this information, compute $\binom{3}{2}$, and $\binom{4}{2}$.

11. In fact, with this new information, we can compute $\binom{n}{k}$ for any $n$ and any $k$. Let’s try $\binom{17}{7}$, but stop before you do any multiplication.

Putting together what we have learned today, we have that

$$\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n-1} + \binom{n}{n} = 2^n.$$ 

Today, we have learned about the Power Set, that there are $2^n$ elements in the power set for a set of $n$ elements, and what choose and factorial notation are.