Overview

In this two (or three) day set of lessons, students will explore the concept of a power set of a set X, which is the set of all subsets of the set X. For example, say that X is a set of paint colors: red, green, and blue. Given X, how many color combinations are there? First there is combination with no colors, we call this subset of X the **empty set**. Then we have three subsets of X comprised of exactly one color: red on its own, blue on its own, and green on its own. There are also three subsets of X comprised of exactly two colors: red and green together, red and blue together, and green and blue together. Finally there is the subset of X that is X itself, that is the set of all three colors: red, green, and blue. The total number of subsets of X is $8 = 1 + 3 + 3 + 1$. Since the elements of the power set of X are the subsets of X, then there are 8 elements of the power set of X.

One can come up with many examples of power sets, such as the number of possible sundae toppings, or the number of possible committees given a set of people.

Students will first participate in a hands-on discovery activity on Day One. Then on Day Two, students will formalize their findings from the activity on Day One, through a question and answer lecture session. There is a bonus session where students can use the skills and notation from the previous two days into exploring an extension of the power set.

Math Standards


Focus Question

For any number of light switches in a row, can you write down a formula for the number of unique ways to switch on and off the light switches? For example, if they have two light switches, one way is to have Light Switch 1 on and Light Switch 2 off. In this activity, this way is not the same as Light Switch 1 off and Light Switch 2 on. In other words, the location of the light switch does matter.

Objectives

Through this lesson, students will:

1. Define a power set and give at least two concrete examples
2. Articulate why there are $2^n$ elements in the power set of a set with $n$ elements.
3. State what $\binom{n}{k}$ means in words and will be able to compute it.
4. State what a factorial is and will be able to compute it.
Background

Students should be comfortable with multiplication and fractions to be successful in this activity. There are many opportunities for students to strengthen their skills surrounding both in this activity.

The idea of the power set is a very natural one: it is simply the number of possible subsets of a given set $X$ (note: for this activity, we only consider finite sets). The two sticking points that some have with the power set are 1) the idea of the empty set, which is the set of no elements in $X$, and 2) that $X$ is a subset of itself. Thinking about paint colors or sundae toppings can help alleviate both of those sticking points.

One idea used throughout Day Two is the idea of a combination: given $n$ distinct elements, how many ways can you choose $k$ of them. We write $\binom{n}{k}$ and say “$n$ choose $k$.” For $\binom{n}{0}$ and say “$n$ choose 0” meaning “how many ways can we choose none of the $n$ objects?” Consider the set of paint colors from above: red, green, and blue. We have already shown that $\binom{3}{2} = 3$, by writing out the list of subsets of $X$ comprised of exactly two colors: red and green together, red and blue together, and green and blue together. Some students may insist that red and blue is not the same subset of $X$ as blue and red, because of the order that the colors appear in. For combinations, we do not care what order the colors appear in.

Permutations, which are similar to combinations, do care about the order of the colors. For example, say we have a deck of cards that we draw one card at a time from. If we want to know how many ways we can first draw a king and then a 6, then we would use permutations. However, we want to know how many hands of two cards contain a king and a six, we would use combinations, because we do not care about the order that the cards are drawn in (king then 6, or 6 then king), just the result after two draws.

A pleasant fact (which for many is surprising) is that the number of elements in the power set of $X$ with $n$ elements is equal to the following sum: $\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n}$.

The following Wikipedia pages are about power sets, combinations, and permutations. These pages cover a fair amount of ground in many areas of mathematics and can be daunting, but they can serve as a good beginning point for more curious students.

Combinations: http://en.wikipedia.org/wiki/Combination
Permutations: http://en.wikipedia.org/wiki/Permutation

Materials

Teacher Handouts: PowerSet – Overall outline for 2 days of activities, plus bonus activity.
LessonPlan_Day2 – Lesson plan for Day 2
HW_Answers_Day2 – Homework answers for Homework assigned after Day 2

Activity Sheets: LightSwitches – Activity for Day 1
StarsCandyBars – Activity for Day 1
Boxes, Stars – Handouts needed for StarsCandyBars Activity
Worksheet_Day2 – Worksheet for Lesson on Day 2
HW_Day2 – Homework assigned after Day 2

Candy bars in mini-size will be needed for Day 1 Activities and for the bonus activity.

Bonus Day: Group Assessment – A group exploration moving beyond the power set
Boxes, Stars – Handouts needed for Group Assessment Activity
Group Assessment Answers – Answers for Group Assessment
Preparation

Read through the PowerSet document before implementing these activities. Print enough Activity sheets for each person in the class as well as enough Boxes and Stars sheets for each group of 3 to 4 students. Buy mini-candy bars for Day 1 and Bonus day activities.

Procedure

The procedure is also outlined in PowerSet document with details from some of the worksheets included.

**Day One:** Hands-on Exploration of the Power Set
1) Form groups comprised of 3 to 4 students. If possible, try to form groups comprised of students with different approaches to solving problems.
2) Divide the groups into two different kinds: A and B.
3) Give A groups Activity 1: LightSwitches and give B groups Activity 2: StarsCandyBars, and give groups the necessary associated components.
4) Allow student about 35 minutes for their respective activities.
5) Now give A groups Activity 2: StarsCandyBars and give B groups Activity 1: LightSwitches, and give groups the necessary associated components.
6) Allow student about 15 minutes for their respective activities.
7) Bring the class all together for the following discussion questions:
   a. How did you approach these tasks?
   b. Are these two tasks different or the same? Why or why not?
   c. What if instead of 2 choices for each switch or box, you had 3 choices? What about 4 choices?

**Day Two:** Formalization of ideas from Day One
1) Unlike Day One, Day Two is a question and answer session altogether as one class. The document LessonPlan_Day2 is the lesson plan for the day. This lesson was implemented in a class where the person giving the lesson did not know the students, employed the technique “think-pair-share” for sets of questions, and cold-called on students for answers, so there are parts of the lesson that include wording that is specific to that situation. Words in italics are notes to the presenter and not meant to be said out loud.
2) There is a worksheet called Worksheet_Day2 for the students to record answers on and follows the lesson laid out in LessonPlan_Day2. Most of the answers for Worksheet_Day2 are in LessonPlan_Day2.
3) At the end of the lesson, give students HW_Day2 for homework to help cement the material that they have been introduced to.

**Bonus Day:** Assess understanding of concepts as well as explore further extensions of the power set.
*Note: this session is entirely optional below are suggestions for how to proceed.*
1) Before class, write an Individual part, suitable for your class to do in no more than 15 minutes, that checks students’ understanding of the basic definitions: power set, combination, and factorials.
2) Give the individual part, allowing no more than 15 minutes.
3) Form groups comprised of 3 students. If possible, try to form groups comprised of students with different approaches to solving problems.
4) Give each group the activity called Group Assessment, and give groups the necessary associated components.
5) Allow groups at least 35 minutes to work on this activity.
Assessment

The document *HW_Day2* can be used as a homework assignment or as an assessment. Alternatively, one can use the bonus day activities as an assessment to see if the students understood the mechanics behind their power set exploration.

Extensions

Beyond the bonus day activity, one can explore concepts that make use of combinations and permutations, such as the binomial theorem, binomial coefficients, and Pascal’s triangle. Pascal’s triangle is especially nice because there is a nice hands-on way to build Pascal’s triangle, which can then be connected to some facts about combinations, the binomial theorem, and the binomial coefficients.

The following Wikipedia pages are about binomial theorem, binomial coefficients, and Pascal’s triangle, and contain a good amount of fun mathematical facts about all three as well as great visuals and applications.

- Pascal’s triangle: [http://en.wikipedia.org/wiki/Pascal%27s_triangle](http://en.wikipedia.org/wiki/Pascal%27s_triangle)