Measuring Upward Mobility

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August 7, 2012

Abstract

This paper develops a mobility measure that combines desirable features of the relative and absolute approaches to measuring mobility. From the relative approach we borrow the idea that income growth accruing to the relatively poor is more to be valued than income growth to the relatively rich. From the absolute approach we take the idea that upward changes are to be valued, in and of themselves. Our combined approach precipitates (under some axioms) a class of mobility measures, fully characterized up to a single parameter.

JEL: [Poverty and Inequality] I32, O15

1. Introduction

In its broadest sense, mobility refers to the ease of transition between various economic and occupational categories in the society. Our goal in this paper is to provide a measure of this ease of transition. However, the mobility that we are really interested in is "upward mobility". Loosely speaking, we would like to say that a society is "mobile" if individuals — especially the relatively poor — can put their historical disadvantages behind and enjoy the overall gains of society. The greater is their share of the gains, the more mobile the society. This is an imprecise description but even at this level of imprecision two important ethical underpinnings of the term become very clear. First, by mobility we want to measure the ease of *gains*, rather than the ease of *losses*. Second, we wish to place higher weight on the gains of the relatively poor.

A large literature on mobility is built around the idea that transition probabilities across *relative* income (or "status") rankings are all we need to know for a suitable measure.[1] Thus suppose that $r \in [0, 1]$ is an index of relative position in the income distribution, where 0 is the lowest position and 1 the highest. In actual empirical situations, r could stand for quintiles or deciles or some other index of relative status. Panel data on r would be available for two or more periods of time. Under this approach, a mobility measure is some function of the joint distribution of r across time. Measuring this sort of mobility has attracted much attention in the economics literature; see, for example, Atkinson [**?**], Bartholomew [**?**], Chakravarty, Dutta and Weymark [**?**], Conlisk [**?**], Dardanoni [**?**], Prais [**?**] or Shorrocks [**?**] and [**?**]).

The conceptual problem that arises with this approach is that it divests itself of much of the ethical connotations that swirl around the word "mobility". It is impossible, in particular, to separate absolute gains from absolute losses, and in some approaches such "relative mobility" is not readily distinguishable from volatility. A good example is Shorrocks [**?**], in which an increase in some off-diagonal entry of the transition matrix is presumed to be mobility-enhancing, without particular concern about whether someone is gaining or losing by this change. One way to defend this approach is to argue that ultimately, a measure of mobility must be combined with summary statistics, such as inequality or the overall rate of growth, to come to an overall welfare assesment. But there is no reason to suppose that our final welfare judgments are representable in this two-step procedure. It may be necessary, *at the outset*, to combine mobility information with information about which groups experience that mobility, as well as keep track of whether that mobility is "upwards" or "downwards".

Fortunately, there are variants of the "relative mobility" approach that are concerned directly about who gains and who loses. For instance, the important contribution by Chakravarty, Dutta and Weymark [**?**] explicitly compares income distributions before and after, and their mobility measure rewards equalizations (see also the closely related notion of an "equalization measure" in Fields [**?**], as well as the notion of "equality of opportunity" discussed by Bènabou and Ok [**?**]).

[1] This notion is sometimes called "pure mobility" or "exchange mobility" in the sociological literature (see Dardanoni [**?**] and Markandya [**?**]).

A more fundamental problem with the relativistic approach is that all "balanced" growth that preserves relative positions is simply normalized away. Of course, one could simply "add it back" to get an overall welfare assessment but the problem is more complicated than that: there could be "unbalanced" growth that necessitates the making of a conceptual tradeoff across overall mobility and (dis)equalizing mobility. Such tradeoffs have to be dealt with from an explicit axiomatic perspective rather than in an *ad hoc* way.

In short, apart from the question of loser and gainer identity, there is no getting away from the "absolute" aspects of mobility. Significant contributions to the concept of absolute mobility include Fields and Ok [**?**] and Mitra and Ok [**?**]. These papers explicitly permit overall income changes to be recorded as enhancing mobility. Indeed, they demand it (see, for instance, Axiom 2.1 in Fields and Ok [**?**] and Axiom LH in Mitra and Ok [**?**]). However, the main axiomatic results in these papers precipitate a measure of mobility that simply depends on the magnitude (but not the direction) of change in incomes. In this sense, these are measures of "movement" and not of "mobility", with all its attendant ethical connotations. An exception is Fields and Ok [**?**], which discusses "directional mobility measures" that sum the individual income growth rates.[2], but the act of summation throws away information about who gains and who loses. As we've already mentioned, the relative mobility literature contains contributions that are sensitive to this issue.

This paper suggests a measure of mobility that combines (in our view) the two positive aspects of the relative and absolute approaches to measuring mobility.

From the relative approach we borrow the idea that income growth accruing to the relatively poor is more to be valued than income growth to the relatively rich. From the absolute approach we take the idea that upward changes are to be valued, in and of themselves. It turns out that our combined approach precipitates (under some explicit axioms) a sharp class of mobility measures, one that is described via the variation of just a single parameter. It is distinct from other measures of mobility that have been studied in the literature but closely related to measures of pro-poor growth that have been studied in a different context.

What we are after, then, is a measure of economic mobility that both welcomes an increase in income and also rewards situations in which poorer income groups growing faster. It is clear from our brief review of the literature that each of these aspects can be found in some paper, so we are obviously not the first to think of any of these points. Moreover, it is very simple to combine existing measures or add to them in a way that satisfy both desiderata. As an example, adjust the Chakravarty-Dutta-Weymark measure by adding to it (or multiplying it by) some aggregative income indicator. Or see Silber and Weber [**?**], in which the authors write down three mobility measures, based on the Gini coefficient, which are sensitive both to aggregate income changes as well as transfers of income between relatively rich and relatively poor.

[2]We will discuss their approach in some more detail below.

The problem with these variations is that a clear and rigorous axiomatic foundation is missing, and it would be impossible to evaluate the variations from some compelling ethical perspective. Certainly, the axiomatic foundations are themselves open to question, but at the very least the researcher interested in applying a particular measure would know where she stands if the measure is developed from such a foundation. An example of this sort of exercise is provided at the end of this section, where we use our axioms to evaluate the "directional mobility" measure developed by Fields and Ok [**?**].

We now proceed to an axiomatic treatment.

2. Measuring Upward Mobility

We derive a measure of economic mobility that satisfies two fundamental desiderata:

1. The measure should be "directional" and not just based on "movement": it should reward higher growth.

2. The measure should be sensitive to the relatively poor: it should reward any "transfer of growth" from higher to lower income groups.

Suppose that there are n individuals, with the income of person i given by y_i. Assume that $n \geq 3$. For each person i we observe, in addition, a growth rate g_i, which may be positive or negative. Notice that if the time period is small (say time is continuous) then these growth factors cannot result in a "crossing" of individuals by income "in the next period". Later, we extend the measure to handle crossings.

Let $\mathbf{z}_i = (y_i, g_i)$, where $y_i > 0$ and $g_i \geq 0$ is any real number. Denote by $\mathbf{z}$ the full collection $(z_1; \ldots; z_n)$, and by $\mathbf{Z}$ the space of all conceivable $\mathbf{z}$'s. A mobility measure is a smooth real-valued mapping M on $\mathbf{Z}$,[3] which is symmetric or anonymous with respect to permutations of individual names. The following axioms are imposed on M:

[M.1] (Sensitivity to Growth) If $g_i \geq g'_i$ for all i (with strict inequality for some i), then $M(\mathbf{z}) > M(\mathbf{z}')$.

[M.2] (Balanced Growth Normalization) If $g_i = g$ for all i, then $M(\mathbf{z}) = g$.

[M.3] (Sensitivity to the Poor) Suppose that $y_i < y_j$. Then $M(\mathbf{z}') > M(\mathbf{z})$, where $\mathbf{z}'$ is exactly the same as $\mathbf{z}$ but has $g'_i = g_i + \epsilon$ and $g'_j = g_j - \epsilon$ for some $\epsilon > 0$.

[M.4] (Irrelevant Alternatives For Growth Tradeoffs) Comparisons between the growth rates of any two individuals are unaffected by other individuals. Formally, consider i and j with incomes y_i and y_j, and let (g_i, g_j) and (g'_i, g'_j) be two alternative pairs of growth rates. Let $z_k = (y_k, g_k)$ and $z'_k = (y_k, g'_k)$ for $k = i, j$. Denote the incomes and growth rates of everyone else by $\mathbf{z}_{-ij}$. Then the mobility ranking of

$$\mathbf{z} = (z_i, z_j, \mathbf{z}_{-ij})$$

[3] We could have just as easily started with an ordering on $\mathbf{Z}$ and derived a representation under completely standard axioms.

and

$$\mathbf{z}' = (z'_i, z'_j, \mathbf{z}_{-ij})$$

is independent of the specific value of $\mathbf{z}_{-ij}$.

[M.5] (Relativity of Baseline Incomes) If $M(\mathbf{z}_1) \geq M(\mathbf{z}_2)$, then $M(\mathbf{z}'_1) \geq M(\mathbf{z}'_2)$, where $\mathbf{z}'_1$ and $\mathbf{z}'_2$ scale all incomes in $\mathbf{z}_1$ and $\mathbf{z}_2$ by exactly the same constant, but leave all growth rates unchanged.

Remark 1. Notice that [M.1] and [M.2] does not necessarily imply that all *aggregate* growth is welcomed. Imagine switching a growth rate of 5% (for the poor) and 10% (for the rich) between the poor and the rich. Then [M.3] states that mobility must rise, while aggregate growth is lower.

Remark 2. [M.3] states that *no matter* what the growth rates of rich and poor are, a transfer of that growth from rich to poor increases mobility. The axiom makes sense provided that there are no income crossings. (This proviso is indeed satisfied in a continuous time setting.)

Proposition 1. *A mobility measure satisfies* [M.1]–[M.5] *if and only if it can be written as*

$$M_\alpha(\mathbf{z}) = \frac{\sum_{i=1}^n g_i y_i^{-\alpha}}{\sum_{i=1}^n y_i^{-\alpha}} \tag{1}$$

for some $\alpha > 0$.

Proof. The representation (1) satisfies all the conditions [M.1]–[M.5], so the nontrivial direction is "necessity".

Step 1. By the same argument as Gorman [**?**], $n \geq 3$ and axiom [M.4] forces the existence of a separable representation (in the g's):

$$M(\mathbf{z}) = \sum_{i=1}^n m(y_i, \mathbf{y}, g_i),$$

where (by symmetry) m is the same function for all individuals and $\mathbf{y}$ is the vector of all incomes.

Step 2. By [M.1], $m(y, \mathbf{y}, g)$ is an increasing function of g, and by [M.2], $m(y, \mathbf{y}, 0) = 0$ for all y and $\mathbf{y}$. Also m is differentiable because M is. Denote by m' the derivative of m with respect to its last argument, g.

Step 3. We claim that there exist differentiable functions π and h, where $\pi(\mathbf{y}) > 0$ for all y and $h(y, g)$ is an increasing function of g with $h(y, g) = 0$ for all y, such that for all individuals i,

$$m(y_i, \mathbf{y}, g_i) = \pi(\mathbf{y}) h(y_i, g_i) \tag{2}$$

To prove this claim, we first show that

$$m'(y_i, \mathbf{y}, g_i) = \psi(z_i, z_j) m'(y_j, \mathbf{y}, g_j) \tag{3}$$

for some function ψ that depends only on z_i and z_j. That is, the ratio of the derivatives is independent of all data not pertaining to i and j. To see this, note that the local mobility tradeoff between i and j that keeps mobility constant is given by the relative values of $m'(y_i, \mathbf{y}, g_i)$ and $m'(y_j, \mathbf{y}, g_j)$, so by [M.4], this must be independent of all data not involving i and j.

Now integrate (3) over the g_i-dimension to see that

$$\begin{aligned} m(y_i, \mathbf{y}, g_i) &= m'(y_j, \mathbf{y}, g_j) \int_0^{g_i} \psi(y_i, x, z_j) dx + m(y_i, \mathbf{y}, 0) \\ &= m'(y_j, \mathbf{y}, g_j) \int_0^{g_i} \psi(y_i, x, z_j) dx, \end{aligned}$$

where we use Step 2. For any $g_i \neq 0$, the integral in this expression is non-zero. It follows that for $g_i \neq 0$,

$$m'(y_j, \mathbf{y}, g_j) = \hat{\psi}(z_i, z_j) m(y_i, \mathbf{y}, g_i),$$

where $\hat{\psi}(z_i, z_j) \equiv \left(\int_0^{g_i} \psi(y_i, x, z_j) dx\right)^{-1}$. Integrating this time over the g_j-dimension,

$$\begin{aligned} m(y_j, \mathbf{y}, g_j) &= m(y_i, \mathbf{y}, g_i) \int_0^{g_j} \hat{\psi}(z_i, y_j, x) dx + m(y_j, \mathbf{y}, 0) \\ &= m(y_i, \mathbf{y}, g_i) \Psi(z_i, z_j), \end{aligned} \tag{4}$$

where once again we use Step 2 and define $\Psi(z_i, z_j) \equiv \int_0^{g_j} \hat{\psi}(z_i, y_j, x) dx$. Because (4) is true over all $\mathbf{y}$, this must imply

$$m(y_i, \mathbf{y}, g_i) = \pi^{ij}(\mathbf{y}) h^{ij}(y_i, y_j, g_i)$$

for some functions π^{ij} and h^{ij}. But there are at least three individuals, so the above equation also holds with j replaced by k. This means that (2) must be true. Note well that by symmetry, π and h are *independent* of the group.

STEP 4. We claim that h must be linear in g for each y. Suppose not; then there exists some value y_1 for which $h(y_1, g)$ has two different slopes in g at two different values of g. More formally, there exist y_1, g_1, g_2 and $\epsilon > 0$ such that

$$h(y_1, g_1 + \epsilon) - h(y_1, g_1) < h(y_1, g_2) - h(y_1, g_2 - \epsilon). \tag{5}$$

Take a (nonconstant) sequence of incomes converging to y_1 from above. Let y_2 be a typical term in that sequence; then $y_2 > y_1$. Construct a scenario in which $z_1 = (y_1, g_1)$, $z_2 = (y_2, g_2)$, and z_i is some arbitrary specification for all other i. Next, define $z_1' = (y_1, g_1 + \epsilon)$ and $z_2' = (y_2, g_2 - \epsilon)$. Let $\mathbf{z}$ be the collection $(z_1, z_2, z_3, \ldots, z_n)$, and $\mathbf{z}'$ the collection $(z_1', z_2', z_3, \ldots, z_n)$. Note that

$$\frac{M(\mathbf{z}') - M(\mathbf{z})}{\pi(\mathbf{y})} = \{[h(y_1, g_1 + \epsilon) - h(y_1, g_1)] - [h(y_2, g_2) - h(y_2, g_2 - \epsilon)]\}.$$

At the same time, as $y_2 \downarrow y_1$, we know by continuity that $h(y_2, g_2) - h(y_2, g_2 - \epsilon) \to h(y_1, g_2) - h(y_1, g_2 - \epsilon)$. Using (5), we must therefore conclude that for y_2 far enough in the sequence, $M(\mathbf{z}') - M(\mathbf{z}) < 0$. But this contradicts [M.3].

STEP 5. We may therefore write $h(y,g)$ as $\phi(y)g$, where (by [M.1]), it must be that $\phi(y) > 0$ for all y. We claim next that

$$\phi(y) = y^{-\alpha} \quad \text{for some } \alpha > 0. \tag{6}$$

To prove this, we first show that for every $(y_1, y_2) \gg 0$ and every $\lambda > 0$,

$$\frac{\phi(y_1)}{\phi(y_2)} = \frac{\phi(\lambda y_1)}{\phi(\lambda y_2)}. \tag{7}$$

Suppose that this is false for some $(y_1, y_2) \gg 0$ and $\lambda > 0$. Without loss of generality suppose that $y_1 < y_2$ and that "$>$" holds in the inequality above. Pick values g_1, g_1', g_2, g_2' such that $g_1 > g_1'$ and $g_2' > g_2$, and such that

$$\frac{\phi(y_1)}{\phi(y_2)} > \frac{g_2' - g_2}{g_1 - g_1'} > \frac{\phi(\lambda y_1)}{\phi(\lambda y_2)}. \tag{8}$$

Now consider two situations $\mathbf{z}$ and $\mathbf{z}'$. Under $\mathbf{z}$, the values for persons 1 and 2 are (y_1, g_1) and (y_2, g_2), while under $\mathbf{z}'$, the corresponding values are (y_1, g_1') and (y_2, g_2'). For all other individuals, $\mathbf{z}$ and $\mathbf{z}'$ have identical specifications. Manipulating the left inequality in (8), we must conclude that

$$\phi(y_1)g_1 + \phi(y_2)g_2 > \phi(y_1)g_1' + \phi(y_2)g_2',$$

and consequently, that $M(\mathbf{z}) > M(\mathbf{z}')$. Now scale every income in $\mathbf{z}$ and $\mathbf{z}'$ by the common factor λ (given above) and call the new situations $\mathbf{z}_\lambda$ and $\mathbf{z}'_\lambda$. Manipulating the right inequality in (8), we must conclude that

$$\phi(\lambda y_1)g_1 + \phi(\lambda y_2)g_2 < \phi(\lambda y_1)g_1' + \phi(\lambda y_2)g_2',$$

so that now we have $M(\mathbf{z}'_\lambda) > M(\mathbf{z}_\lambda)$. But this reversal contradicts [M.5]. Therefore (7) must be true.

By defining $y = y_1$, $y' = y_2/y_1$, and $\lambda = 1/y$, we see from (7) that ϕ satisfies the fundamental Cauchy equation

$$\phi(y)\phi(y') = \phi(yy')\phi(1) \tag{9}$$

for every $(y, y') \gg 0$. The class of solutions to (9) (that also satisfy continuity and $\phi(y) > 0$ for $y > 0$) must be proportional to $\phi(p) = p^{-\alpha}$ for some constant α (see, e.g., Aczél [1966, p. 41, Theorem 3]). Invoking [M.1] again, it is easy to see that α must be positive.

We have therefore shown that

$$M(\mathbf{z}) = \phi(\mathbf{y}) \sum_{i=1}^{n} g_i y_i^{-\alpha}$$

for some $\alpha > 0$. Now use the normalization [M.2] to conclude that $\phi(\mathbf{y}) = \left(\sum_{i=1}^n y_i^{-\alpha}\right)^{-1}$, which completes the proof. ■

The parameter α captures the *weight* that the measure places on the poor. If α is close to 0 then our mobility measure (1) is approximately the sum of the growth rates of individual income, and as $\alpha \to \infty$ the measure becomes approximately Rawlsian.

We make two remarks. First, our mobility measure contributes to the literature on *pro-poor growth.* Chenery et al. [**?**] suggested the evaluation of economic performances as the weighted sum of the growth rates of income of all economic groups (see Chapter 2 in this book, by Ahluwalia and Chenery, as well as Ravallion and Chen [**?**] and Essama-Nssah [**?**] for a discussion of specific weights). Our axiomatic approach suggests a set of weights based on the inverse of individual's initial income that may be used when assessing pro-poor growth.

Second, we note that Fields and Ok [**?**] also provide an axiomatic derivation for a mobility measure that (a) rewards growth and (b) is sensitive to inequality. Without going into detail about the setting or the axioms, we simply record the measure that they obtain:[4]

$$M^{FO}(z) = \frac{1}{n}\sum_{i=1}^{n} \ln g_i = \frac{1}{n}\sum_{i=1}^{n}\left[\ln(y_i') - \ln(y_i)\right],$$

where y_i is starting income and y_i' is ending income over two periods. While Fields and Ok use different axioms in a different setting, we use our axioms to evaluate this measure.

Notice that this measure is sensitive to growth, so that [M.1] is satisfied. It also satisfies [M.2], in the sense that if all growth rates are the same, say g, the measure yields a monotone transform of g. Axioms [M.4] and [M.5] are satisfied as well, the latter trivially so because baseline income weights do not even appear in the measure. We must conclude, therefore, that their measure cannot satisfy Axiom [M.3], which asserts the sensitivity of mobility to pro-poor growth.

The following example demonstrates this directly. Suppose that there are just two individuals (or equally-sized groups), with incomes 100 and 200, and suppose that both grow at 10%. Compare this with another situation in which the poor group grows at 15%, while the rich grow at 5%. Growth is now pro-poor, and our mobility measure goes up. The measure M^{FO}, however, comes *down.*[5] Axiom [M.3] is violated. That is not to say that the measure we propose here is unambiguously "better", but only to observe that the researcher has an explicit ethical basis on which to compare M and M^{FO}.[6]

Notice, however, that their measure is concave in ending incomes, and so is sensitive to an improvement in equality at the terminal date. But this has to do with equality in income *levels.* Our mobility measure satisfies this condition as well, but also rewards

[4]They call this a "directional measure" to emphasize that "upward" changes in income are preferred to downward changes; as already discussed, their other measures no not have this property.

[5]The logarithm is a strictly concave function, so a mean-preserving spread of the growth rates must bring the average value down.

[6]The fact that one of our axioms is violated is neither here nor there. Certainly, our measure violates at least one of the Fields-Ok axioms as well! The point is that the axiomatic approach makes for clarity in evaluation.

(positive) differential growth for the poor. The measure M^{FO} may punish such growth, as the example above shows.

3. Measuring Mobility over a Discrete Period

The linearity of our mobility measure in growth rates is largely a consequence of [M.3], which to our knowledge is new in the literature. At first glance, it looks like a Pigou-Dalton transfer axiom and in a sense it is, but it is a requirement placed on growth *rates* rather than on income *levels*. It is a consequence of the presumption that in very short time periods, differential growth rates cannot generate a crossing of incomes between rich and poor. Can we therefore be confident that the mobility measure can be suitably applied across discrete time periods (say, generations), in which some crossings could surely have occurred? After all, in practice, data can only be collected at discrete intervals of time.

In this section, we extend our mobility measure to handle such cases.

Consider two individuals, A and B, with incomes (x_A, x_B) at the beginning of the period and (y_A, y_B) at the end. Initially, A is poorer than B but her income grows fast and she quickly outperforms B and ends at a higher income. Figure 1 illustrates. By converting end incomes into growth rates and mechanically applying the measure (1), we would place more weight on A's growth than on B's growth, despite the fact that A is richer than B for most of the intervening period. Rather, one would want to favor A's growth as long as she is poorer but put more weight on B's growth once he becomes poorer.

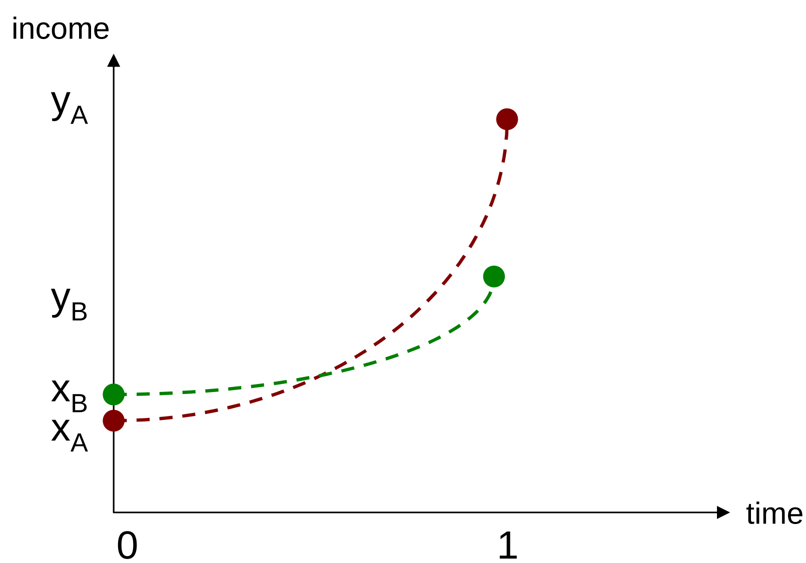

Figure 1. Income Crossing.

One way to implement this idea is to use the implied continuous time growth rate to "interpolate" the income trajectory of individuals in between the two discretely separated observations. Normalize the time period of observation to 1; let 0 represent the initial date. Denote by $\mathbf{x} \equiv \{x_1, x_2, ...x_n\}$ the vector of incomes at time 0 and by $\mathbf{y} \equiv \{y_1, y_2, ...y_n\}$ the vector of incomes at time 1.

The constant interpolated growth factor for individual i is given by $\gamma_i \equiv \ln(y_i) - \ln(x_i)$. We therefore approximate the income of individual i at any time $t \in [0, 1]$ by

$$y_i(t) = x_i e^{\gamma_i t}.$$

Let $\mathbf{z}(t)$ be the resulting distribution of income and growth rates $\{y_i(t), \gamma_i\}$ at time $t \in [0, 1]$. We apply the measure in (1) to $\mathbf{z}(t)$ for every $t \in [0, 1]$ and average it over the period. This gives us the *discrete mobility measure*

$$\mathcal{M}(\mathbf{x}, \mathbf{y}) \equiv \int_0^1 M(\mathbf{z}(t))dt.$$

Proposition 2. *A discrete mobility measure can be written as*

$$\mathcal{M}(\mathbf{x}, \mathbf{y}) = \ln \left[\frac{\sum_i y_i^{-\alpha}}{\sum_i x_i^{-\alpha}} \right]^{-\frac{1}{\alpha}} \tag{10}$$

for some $\alpha > 0$.

Proof. Applying our mobility measure at each instant in time, we see that

$$\begin{aligned}
\mathcal{M}(\mathbf{x}, \mathbf{y}) &= \int_0^1 \frac{y_i(t)^{-\alpha} \gamma_i}{\sum_i y_i(t)^{-\alpha}} \\
&= \int_0^1 \frac{x_i^{-\alpha} e^{-\alpha \gamma_i t} \gamma_i}{\sum_i x_i^{-\alpha} e^{-\alpha \gamma_i t}} \\
&= -\frac{1}{\alpha} \left[\ln \left(\sum_i x_i^{-\alpha} e^{-\alpha \gamma_i t} \right) \right]_{t=0}^{t=1} \\
&= -\frac{1}{\alpha} \ln \left[\frac{\sum_i x_i^{-\alpha} e^{-\alpha \gamma_i}}{\sum_i x_i^{-\alpha}} \right] \\
&= \ln \left[\frac{\sum_i y_i^{-\alpha}}{\sum_i x_i^{-\alpha}} \right]^{-\frac{1}{\alpha}}.
\end{aligned}$$

■

This measure satisfies the main properties that, as we discussed, are deemed desirable for a mobility measure:

[1] Higher growth for any person increases mobility, *ceteris paribus*. The measure is clearly increasing in y_i, for every i.

[2] If all the growth factors are the same — say, $y_i = \lambda x_i$ for every i for some $\lambda > 0$, then $\mathcal{M}(\mathbf{x}, \mathbf{y}) = \ln \lambda$, which is exactly the common implicit continuous-time growth rate.

[3] An increase in an individual growth rate increases mobility more the poorer that person is. To see this, note that the discrete mobility measure can be written as

$$\mathcal{M}(\mathbf{x}, \boldsymbol{\gamma}) = -\frac{1}{\alpha} \ln \left[\sum_i x_i^{-\alpha} e^{-\alpha \gamma_i} \right] + \frac{1}{\alpha} \ln \left[\sum_i x_i^{-\alpha} \right]$$

where (with a slight abuse of notation) we change the domain to baseline incomes and subsequent growth rates, just as in the instantaneous version. Differentiating this expression first with respect to some γ_k and then with respect to x_k, we see that

$$\frac{\partial^2 \mathcal{M}(\mathbf{x}, \boldsymbol{\gamma})}{\partial \gamma_k \partial x_k} = -\alpha \frac{x_k^{-\alpha-1} e^{-\alpha \gamma_k} \sum_{i \neq k} x_i^{-\alpha-1} e^{-\alpha \gamma_i}}{(\sum_j x_i^{-\alpha-1} e^{-\alpha \gamma_i})^2} < 0.$$

[4] Multiplying all incomes by the same constant does not affect the discrete mobility measure.

[5] $\mathcal{M}$ satisfies the population neutrality principle: increasing the population size by replicating the existing society (the distribution of initial income and growth experience) will not affect the mobility measure.

[6] The discrete mobility of a society between period 1 and period T is the sum of the discrete mobility experienced over each period between 1 and T.